MODELS, THEORY AND APPLICATIONS IN COOPERATIVE GAME THEORY

Dongshuang Hou

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PROEFSCHRIFT

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Preface

This thesis consists of an introductory chapter (Chapter 1) followed by nine research chapters (Chapters 2–10), each of which is written as a self-contained journal paper, except that all references are gathered at the end of the thesis. These nine chapters are based on the nine papers that are listed below and have been submitted to journals for publication. The paper that forms the basis for Chapter 5 has been published in the International Journal of Game Theory, the papers underlying Chapter 2 and Chapter 8 are both accepted by the Journal Applied Mathematics and the paper underlying Chapter 10 is accepted by the Journal International Game Theory Review. The other papers are in different stages of the refereeing process. Chapters 2, 3 and 4 deal with game models, Chapters 5, 6 and 7 contain theoretical contributions to Cooperative Game Theory, while Chapters 8, 9 and 10 can be understood as the application of Game Theory. This explains the title of the thesis. Since the thesis has been written as a collection of more or less independent papers, the reader will find a certain amount of repetition of relevant concepts, definitions and background. The author apologizes for any inconvenience.

Papers underlying this thesis

[1] Hou, D. and Theo, T.S.H., (2012), *The Core and Nucleolus in a model of information transferal*, Journal of Applied Mathematics. Article ID 379848, doi:10.1155/2012/379848 (Chapter 2)

[2] Hou, D., Theo, T.S.H. and Aymeric, L., *Convexity and the Shapley value in Bertrand Oligopoly TU-games with Shubik's demand functions.* Working paper (Chapter 3)

[3] Hou, D. and Theo, T.S.H., *The Shapley value and the Nucleolus of Service cost savings games.* Working paper (Chapter 4)

[4] Theo, T.S.H. and Hou, D., (2010), *A note on the Nucleolus for* $2-$*convex TU games.* International Journal of Game Theory. Vol.39 (Chapter 5)

[5] Hou, D. and Theo, T.S.H., (2013), *A new characterization of the Pre-kernel for TU games through its indirect function and its application to determine the Nucleolus for three subclasses of TU games.* Contributions to Game Theory and Management vol. VI(GTM2012) (Chapter 6)

[6] Hou, D. and Theo, T.S.H., *The indirect function of compromise stable TU games as a tool for the determination of its Nucleolus and Pre-kernel.* working paper (Chapter 7)

[7] Hou, D. and Theo, T.S.H., (2012), *Interaction between Dutch Soccer Teams and Fans: A Mathematical Analysis through Cooperative Game Theory.* Journal of Applied Mathematics. Vol.3 No.1 (Chapter 8)

[8] Hou, D. and Theo, T.S.H., *Data cost games as an application of 1-concavity in cooperative game theory.* Working paper (Chapter 9)

[9] Hou, D. and Theo, T.S.H., (2013), *Convexity of the "Airport Profit Game" and k-Convexity of the "Bankruptcy Game"*, accepted by International Game Theory Review (Chapter 10)

Notation

$N = \{1, 2, \cdots, n\}$	the player set
$2^N = \{S \mid S \subseteq N, S \neq \emptyset\}$	the set of all coalitions
$S, S \subseteq N$	coalition
s or $\|S\|$	the cardinality of the set S
G^N	the game space with player set N
TU^N	the cooperative game space with player set N
G	the universe of all game spaces
R	the set of real numbers
R^n	the set of n-dimension vector space
$\vec{e_i}, i \in N$	the i-th unit vector
$v(S)$	the value or worth of coalition S
(N, v)	a cooperative game with transferable utility or TU-game
I^N	imputation set
$C(N, v)$	The Core of game v
$Sh(N, v)$	the Shapley value of game v
$Nu(N, v)$	the Nucleolus of game v

Contents

Chapter 1

Introduction

ABSTRACT - In this chapter, the history of game theory is introduced and a short general introduction to the basis knowledge in game theory is given together with some famous examples.

1.1 Game theory

With the publication of Theory of Games and Economic Behavior by John von Neumann and Oscar Morgenstorn in 1944, cooperative games have been studied for 69 years. Game theory, or interactive decision theory, is a mathematical framework for modeling and analyzing conflict situations involving economic agents with possibly diverging interests. For a given economic problem one extracts the essential features, they are integrated in a model of the game, the game is analyzed and the result is translated back into economic terms. The construction of the appropriate game is not a matter of routine but is an essential part of the analysis. Thus, game theory provides a language and framework allowing for a systematic study of various features of behavioral interaction. It can be used to describe economic situations which at first sight may seem very different and to recognize common elements. Game theory is a mathematical framework that can be classified into two branches: Noncooperative and Cooperative game theory.

Noncooperative game theory can be used to analyze the strategic decision making processes of a number of independent entities, i.e., players, that have partially or totally conflicting interests over the outcome of a decision process which is affected by their actions. Essentially, noncooperative games can be seen as capturing a distributed decision making process that allows the players to optimize, without any coordination or communication, objective functions coupled in the actions of the involved players. We note that the term noncooperative does not always imply that the players do not cooperate, but it means that, any cooperation that arises must be self-enforcing with no communication or coordination of strategic choices among the players.

Cooperative game theory focusses on cooperative behavior by analyzing the negotiation process within a group of players in establishing a contract on a joint plan of activities, including an allocation of the correspondingly generated reward. Particularly, the possible joint reward of each possible coalition (a subgroup of cooperating players) are taken into account in order to allow for a better comparison of each player's role and impact within the group as a whole, and to assign a compromise allocation (a solution) in an objectively reasonable way. Depending on the exact underlying context the coalitional reward can be viewed as the actual result of optimal cooperation or, if partly cooperation is infeasible or if the joint rewards depend on specific assumptions on behavior outside a coalition, as the result of a consistent thought experiment for comparative purposes only. The most basic format of a cooperative game is the model of Transferable Utility games, shortly TU-games. In this thesis, we use TU^N to denote the cooperative game space with player set N.

1.2 Cooperative games in Characteristic Function Form

The following are standard definitions, concepts and theories in Game Theory. We refer to [4] and [33].

Definition 1.1. A cooperative game with transferable utility or TU-game in characteristic function form is an ordered pair (N, v) where N is a finite set and the characteristic function $v : 2^N \to R$ is a characteristic function on the set 2^N of all subsets of N such that $v(\emptyset) = 0$.

A subset S of N is called a coalition. The number $v(S)$ can be regarded as the the value or worth of coalition S in the game v.

Example 1.1. (a glove market game) Let N be divided into two disjoint subgroups L and $R : N = L \cup R, L \cap R = \emptyset$. Members of L each have one left hand glove, members of R one right hand glove. A single glove is worth nothing, a (right-left) pair 10 Euro. This situation can be described by a TU-game (N, v) where

$$v(S) = 10min\{|L \cap S|, |R \cap S|\} \quad \text{for all} \quad S \in 2^N.$$

A TU-game (N, v) is called monotonic if $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subseteq T$. A monotonic game (N, v) with $v(S) \in \{0, 1\}$ for all $S \in 2^N$ and $v(N) = 1$ is called simple game.

Example 1.2. (a voting game) The security council of the United Nations consists of 5 permanent members and 10 nonpermanent members. To pass a resolution, at least 9 (out of 15) member votes to pass are needed, with all 5 permanent members voting to pass the resolution. If we use $T = \{1, 2, ..., 5\}$ to denote the five permanent members and $6, 7, \ldots, 15$ to denote the nonpermanent members, then this voting situation can be described by the simple TU-game (N, v) given by

$$v(S) = \begin{cases} 1, & S \supseteq T, |S| \geq 9; \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the game (N, v) does not reflect monetary gains but voting power instead. A coalition is assigned a value of 1 if and only if this coalition has five permanent members and at least four nonpermanent member votes to pass bills.

Many TU-games (N, v) derived from practical situations satisfy superadditivity, i.e.

$$v(S \cup T) \geq v(S) + v(T) \quad \text{for all} \quad S, T \in 2^N \quad \text{with} \quad S \cap T = \emptyset. \quad (1.2.1)$$

Condition (1.2.1) is satisfied in many of the applications of TU-games. Indeed, it may be argued that if $S \cup T$ forms, its members can decide to act as if S and T had formed separately. Doing so they will receive $v(S) + v(T)$, which implies condition (1.2.1). Nevertheless, quite often superadditivity is violated. Anti-trust laws may exist, which reduce the profits of $S \cup T$, if it forms. Also, large coalitions may be inefficient, because it is more difficult for them to reach agreements on the distribution of their proceeds.

A game (N, v) is called additive if

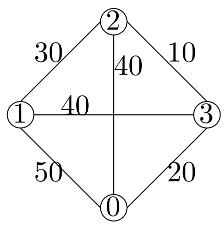
$$v(S \cup T) = v(S) + v(T) \quad \text{for all} \quad S, T \in 2^N \quad \text{with} \quad S \cap T = \emptyset.$$

Note that an additive game is determined by a vector $a \in R^N$ with $a_i = v(\{i\}), i \in N$, since $v(S) = \sum_{i \in S} a_i$ for all $S \in 2^N$. An important notion is strategic equivalence of games: two TU-games (N, v) and (N, w) are called $S-$equivalent if there is a real number $k > 0$ and a vector $a \in R^N$ (an additive game) such that

$$w(S) = k \cdot v(S) + \sum_{i \in S} a_i, \quad \text{for all} \quad S \in 2^N,$$

or shortly, such that $w = kv + a$.
The positive number k reflects a rescaling of monetary units while adding the vector a boils down to giving each player a fixed amount of money (in the new units) independent of the coalition under consideration. Clearly, if v can be used to model a cooperative situation, also w can, and the other way around.

Example 1.3. (A spanning tree game) Consider three communities $1, 2$ and 3 (the players) and a power source 0. For all possible links the connection costs are shown in picture 1.1.



Pic.1.1: The spanning tree problem of Example 1.3.

Assuming that each player has to be connected to the source and the minimal costs of each coalition to connect each of its members to the source is given by the function $c : 2^N \to R$ with

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
c(S)	0	50	40	20	70	60	30	60

Formally, we need to translate the costs into rewards to obtain a TU-game. Obviously this can be done by considering (N, v_0) with $N = \{1,2,3\}$ and $v_0 = -c$. A more standard way to do is to consider the cost savings game (N, v) defined by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S) \quad \text{for all} \quad S \in 2^N.$$

Since $v = v_0 + a$ with $a \in R^N$ such that $a_i = c(\{i\})$ for all $i \in N, v$ and v_0 are $S-$equivalent. For the cost savings game we find

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
v(S)	0	0	0	0	20	10	30	50

We will come back to this type of games later on.

1.3 Solution concepts in Cooperative game theory

This section will make a start with analyzing (solving) TU-games. From now on we are assuming that the players are negotiating about the formation of the grand coalition N and that in the process they are trying to allocate $v(N)$ in a fair and justifiable way among themselves, in particular taking into account the values $v(S)$ of every possible coalition $S \in 2^N$.

Let Φ be a value on G^N where G^N is the game space with player set N. Some obvious requirements of an allocation for a game $v \in TU^N$ are
(i) Efficiency: $\sum_{i \in N} \Phi_i(v) = v(N)$.
(ii) Individual rationality: $\Phi_i(v) \geq v(\{i\})$ for all $i \in N$.
(iii) Linearity: $\Phi(\alpha \cdot v + \beta \cdot w) = \alpha \cdot \Phi(v) + \beta \cdot \Phi(w)$, for all games (N, v), (N, w), and all $\alpha, \beta \in R$.
(iv) Dummy player property: $\Phi_i(v) = v(\{i\})$, for all games and any dummy

player $i \in N$. Player i is a dummy player in the game (N, v) if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus \{i\}$.
(v) Substitution property: $\Phi_i(v) = \Phi_j(v)$, for substitutes i and j in any game. Players i and j are called substitutes if both of them are more desirable, or equivalently, the equality for their marginal contributions, that is, $v(S \cup \{i\}) = v(S \cup \{j\}))$, for all $S \subseteq N \setminus \{i, j\}$.

Allocations satisfying (i) and (ii) are called imputations. The set of all imputations of a game $v \in TU^N$ is denoted by $I(v)$. Clearly we have

$$I(v) \neq \emptyset \Leftrightarrow v(N) \geq \sum_{i \in N} v(\{i\}).$$

Moreover, it is easy to verify that $I(v) = Conv(\{r^i\}_{i \in N})$, where for each $i \in N, r^i \in R^N$ is defined by

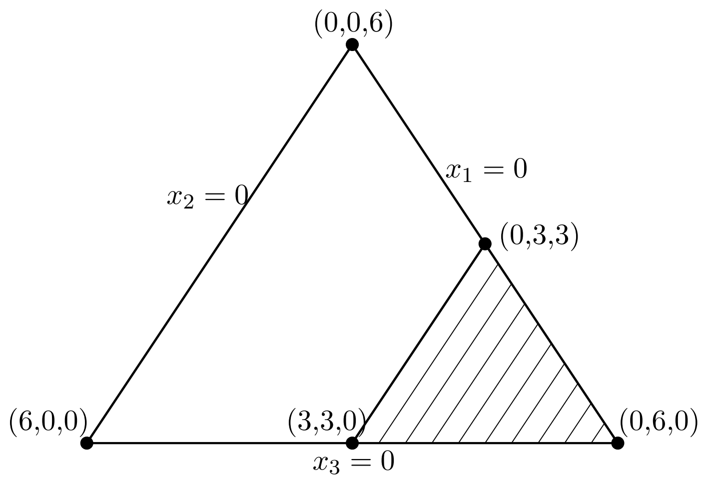
$$r_k^i = \begin{cases} v(\{k\}), & \text{if} \quad k \neq i; \\ v(N) - \sum_{j \in N \setminus \{k\}} v(\{j\}), & \text{if} \quad k = i. \end{cases}$$

for all $k \in N$.

Example 1.4. Let (N, v) be such that $N = \{1, 2, 3\}, v(\{1\}) = v(\{3\}) = 0, v(\{2\}) = 3, v(\{1, 2\}) = v(\{2, 3\}) = 4, v(\{1, 3\}) = 1$ and $v(\{1, 2, 3\}) = 6$. Then $r^1 = (3, 3, 0), r^2 = (0, 6, 0)$ and $r^3 = (0, 3, 3)$. The imputation set $I(v)$ is given by

$$I(v) = Conv(\{(3, 3, 0), (0, 6, 0), (0, 3, 3)\}).$$

The Imputation set of this game can be shown in next picture.



Pic. 1.2: The Imputation set of the game in example 1.4.

Next we introduce one of the most fundamental concepts within the theory of cooperative games.

Definition 1.2. The Core $Core(v)$ of a game $v \in TU^N$ is defined by

$$Core(v) = \{x \in R^n | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } \quad S \subseteq N\}.$$

So, Core elements are imputations (i.e. efficient and individually rational) which are stable against coalitional deviations. No coalition can rightfully object to a proposal $x \in Core(v)$ because what this coalition is allocated in total according to x (i.e. $\sum_{i \in S} x_i$) is at least what it can obtain by splitting off from the grand coalition (i.e. $v(S)$). In particular, if $\sum_{i \in S} x_i > v(S)$, then in any division of $v(S)$ among the members of S, at least one player gets strictly less then what he gets according to x.

Example 1.5. For the game (N, v) of Example 1.4 the Core is given by

$$Core(v) = Conv(\{(2,4,0),(1,5,0),(0,5,1),(0,4,2),(1,3,2),(2,3,1)\}).$$

In general, since the Core is bounded and is determined by a finite system of linear inequalities, it is a polytope: a convex hull of finitely many points. Moreover, it is not difficult to check that the Core is representation-independent. More specifically the Core satisfies relative invariance with respect to $S-$equivalence, i.e. if $w = kv + \alpha (k > 0, \alpha \in R^N)$, then $x \in Core(v)$ implies that $kx + \alpha \in C(w)$.

Unfortunately, a game can have an empty Core. Thus, it is necessary to give an alternative solution. In this sense, many possibilities have been proposed in the literature, as the Shapley value, Nucleolus, the Kernel, Banzhaf value, $\epsilon-$Core, etc.

1.3.1 The Shapley value and convex games

This section introduces the one-point solution concept for TU-games. The Shapley value will assign to each game $v \in TU^N$ a unique vector $Sh(v) \in R^N$. Note that the Core is not a one-point solution concept: the Core may be empty or may consist of more than one point.

The Shapley value $Sh(v) : TU^N \to R^N$ is defined by

$$Sh_i(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v)$$

for all $v \in TU^N$. Here $\Pi(N) := \{\sigma : \{1, 2, \dots, |N|\} \to N \mid \sigma \text{ is bijective}\}$ is the set of all orders on N and the marginal vector $m^\sigma(v) \in R^N$, for $\sigma \in \Pi(N)$, is defined by

$$m^\sigma_{\sigma(k)} = v(\{\sigma(1), \sigma(2), \dots, \sigma(k-1), \sigma(k)\}) - v(\{\sigma(1), \sigma(2), \dots, \sigma(k-1)\})$$

for all $k \in \{1, 2, \dots, n\}$.

In a marginal vector $m^\sigma(v)$ players enter the game one by one in the order $\sigma(1), \sigma(2), \dots, \sigma(n)$ and to each player is assigned the marginal contribution he creates by joining the group of players already present. Since the Shapley value averages all marginal vectors, it thus can be interpreted as an average of marginal contributions.

The Shapley value can also be characterized by means of properties for one-point solution concepts, i.e. efficiency, symmetry, the dummy property and additivity. The combination of these four properties characterizes the Shapley value. Not only does the Shapley satisfy these propertied but it is the only one-point solution concept on TU^N satisfying all four properties at the same time.

Theorem 1.1. *[4] The Shapley value Sh is the unique one-point solution concept on TU^N that satisfies efficiency, symmetry, the dummy property and* *additivity. Moreover, with* $v = \sum_{T \subseteq N, T \neq \emptyset} c_T u_T$*, we have*

$$Sh_i(v) = \sum_{T \subseteq N, T \ni i} \frac{c_T}{|T|}$$

for all $i \in N$.

Example 1.6. Consider the game $v \in TU^N$ with $N = \{1, 2, 3\}, v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1, 2\}) = 20, v(\{1, 3\}) = 10, v(\{2, 3\}) = 30$ and $v(N) = 50$. For this game , all marginal vectors are given by

σ	$m_1^\sigma(v)$	$m_2^\sigma(v)$	$m_3^\sigma(v)$
(1,2,3)	0	20	30
(1,3,2)	0	40	10
(2,1,3)	20	0	30
(2,3,1)	20	0	30
(3,1,2)	10	40	0
(3,2,1)	20	30	0

Since $Sh(v)$ is the average of these six marginal vectors, we find

$$Sh(v) = \frac{1}{6}(0, 20, 30) + \ldots + \frac{1}{6}(20, 30, 0) = (11\frac{2}{3}, 21\frac{2}{3}, 16\frac{2}{3}).$$

Two alternative characterizations of the Shapley value are provided in the next theorem.

Theorem 1.2. *[4] Let* $v \in TU^N$*. Then, for all* $i \in N$*,*

$$(i) Sh_i(v) = \sum_{S \subseteq N, S \not\ni i} \frac{|S|!(n - s - 1)!}{n!}(v(S \cup \{i\}) - v(S)).$$

$$(ii) Sh_i(v) = \frac{1}{n}(v(N) - v(N \backslash \{i\})) + \sum_{j \in N \backslash \{i\}} Sh_i(v, N \backslash \{j\}).$$

Next we provide a characterization of the Shapley value based on the property of strong monotonicity. A solution $\Phi : TU^N \to R^N$ satisfies strong monotonicity if for all games $v, w \in TU^N$ and all $i \in N$ satisfying

$$v(S \cup \{i\}) - v(S) \leq w(S \cup \{i\}) - w(S)$$

for all $S \subseteq N \backslash \{i\}$, it holds that

$$\Phi_i(v) \leq \Phi_i(w).$$

Theorem 1.3. *[49] The Shapley value Sh is the unique one-point solution concept on TU^N that satisfies efficiency, symmetry, and strong monotonicity.*

The Weber set $W(v)$ for a game $v \in TU^N$ is defined as the convex hull of all marginal vectors:

$$W(v) = Conv\{m^\sigma(v) | \sigma \in \Pi(N)\}.$$

Note that $W(v) \neq \emptyset$. The following theorem states that the Weber set contains the Core as a subset.

Theorem 1.4. *[49] Let $v \in TU^N$. Then $Core(v) \subseteq W(v)$.*

Definition 1.3. A game $v \in TU^N$ is called convex if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

for all $S \subseteq T \subseteq N \backslash \{i\}$.

Thus for convex games we have that the greater a coalition is, the greater the marginal contribution is of an individual player joining this coalition. It is easy to check that the spanning tree game of Example 1.3 is convex.

The next theorem provides three alternative characterizations of convex game.

Theorem 1.5. *[49] Let $v \in TU^N$. The following four statements are equivalent:*
(i) *v is convex.*
(ii) *for all $S, T, U \subseteq N$ such that $S \subseteq T \subseteq N \backslash U$*

$$v(S \cup U) - v(S) \leq v(T \cup U) - v(T).$$

(iii) *for all $S, T \subseteq N$*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

(iv) *$Core(v) = W(v)$.*

Condition (*ii*) in Theorem 1.5 allows for an interpretation of convexity in terms of increasing marginal contributions of groups of players (instead of just individuals) when joining larger coalitions.
Condition (*iii*) is a supermodularity condition from which it immediately follows that convex games are superadditive. Condition (*iv*) has many implications. For every TU-games we know that $Core(v) \subseteq W(v)$. Convex games are exactly those games for which there is equality. In particular, since $W(v) \neq \emptyset$ for every $w \in TU^N$, it follows that the Core of convex game is not empty. Moreover, for a convex game v the Shapley value is an element of the Core $Core(v)$. In fact, because $Core(v) = W(v)$ and the Shapley value is the average of all marginal vectors, the Shapley value corresponds to the barycenter of the Core for convex games.

Example 1.7. For the game (N, v) of Example 1.4 it is readily verified that $m^\sigma(v) \in Core(v)$ for all $\sigma \in \Pi(N)$. Hence, by convexity of the Core,

$$W(v) = Conv\{m^\sigma(v) | \sigma \in \Pi(N)\} \subseteq Core(v)$$

and thus $W(v) = Core(v)$. This means that (N, v) is convex game.

1.3.2 The (pre-)Kernel and the (pre-)Nucleolus

To see how (un)happy a coalition S will be with a payoff vector x in a game v, we can look at the excess $e(S, x)$ of S with respect to x defined by

$$e(S, x) = v(S) - x(S).$$

The smaller $e(S, x)$, the happier S will be with x. Note that $x \in Core(v)$ if and only if $e(S, x) \leq 0$ for all $S \subseteq N$ and $e(N, x) = 0$.
If a payoff vector x has been proposed in the game v, player i can compare his position with that of player j by considering the maximum surplus $s_{ij}(x)$ of i against j with respect to x, defined by

$$s_{ij}(x) = \max_{\Gamma_{ij}} e(S, x)$$

where $\Gamma_{ij} = \{S \subseteq 2^N | i \in S, j \notin S\}$.
The maximum surplus of i against j with respect to x can be regarded as the

highest payoff that player i can gain (or the minimal amount that i can lose if $s_{ij}(x)$ is negative) without the cooperation of j. Player i can do this by forming a coalition without j but with other players who are satisfied with their payoff according to x. Therefore, $s_{ij}(s)$ can be regarded as the weight of a possible threat of i against j. If x is an imputation then player j can not be threatened by i or any other player when $x_j = v(\{j\})$ since j can obtain $v(\{j\})$ by operating alone. We say that i outweighs j if

$$x_j > v(\{j\}) \quad \text{and} \quad s_{ij}(x) > s_{ji}(x).$$

The Kernel, introduced by Davis and Maschler in [6], consists of those imputations for which no player outweighs another one.

Definition 1.4. The Kernel $\kappa(v)$ of a game v is defined by

$$\kappa(v) = \{x \in I(v) | s_{ij}(x) \leq s_{ji}(x) \quad \text{or} \quad x_j = v(\{j\}) \quad \text{for all} \quad i, j \in N\}.$$

Definition 1.5. The Pre-kernel $p\kappa(v)$ of a game v is defined by

$$p\kappa(v) = \{x \in R^n | \sum_{i \in N} x_i = v(N), s_{ij}(x) = s_{ji}(x) \quad \text{for all} \quad i, j \in N\}.$$

The Kernel and the Pre-kernel are always non-empty. The Kernel is a subset of the bargaining set. For superadditive games the Kernel and the Pre-kernel coincide.

The subsets of the Pre-kernel and Kernel that belong to the Core coincide.

Theorem 1.6. *[43] For every game (N, v) it holds*

$$p\kappa(N, v) \cap C(N, v) = k(N, v) \cap C(N, v).$$

Also we have the following result about the Pre-kernel and Kernel for convex games.

Theorem 1.7. *[44] When (N, v) is convex, then the Pre-kernel $p\kappa(N, v)$ coincides with the Kernel $K(N, v)$ and consists of only one point.*

Let $I^N = \{v \in TU^N | I(v) \neq \emptyset\}$. For $x, y \in R^n$ we have $x \leq_L y$, i.e. x is lexicographically smaller than (or equal to) y, if $x = y$ or if there exists an $s \in \{1, \ldots, n\}$ such that $x_l \leq y_l$ for all $l \in \{1, 2, \ldots, s-1\}$ and $x_s < y_s$.

Since $e(S, x)$ measures the complaint or dissatisfaction of S with the proposed imputation x, we can try to find a payoff vector which minimizes the maximum excess. We construct a vector $\theta(x)$ by arranging the excesses of the 2^n subsets of N in decreasing order: $\theta_k(x) \geq \theta_{k+1}(x)$ for all $k \in \{1, 2, \ldots, 2^n - 1\}$.

Some important properties of the function θ are summarized below.

Lemma 1.1. *[21] Let $v \in I^N$. Then*
(i) for all $x \in I(v)$, $\theta_1(x) = 0 \Leftrightarrow x \in Core(v)$.
(ii) for all $x, y \in I(v)$ such that $x \neq y$ and $\theta(x) = \theta(y)$, and for all $\alpha \in (0, 1)$,

$$\theta(\alpha x + (1 - \alpha)y) <_L \theta(x).$$

(iii) there exists a unique imputation $x \in I(v)$ such that $\theta(x) \leq_L \theta(y)$ for all $y \in I(v)$.

The unique imputation of Lemma 1.1(iii) is called the Nucleolus. Formally, for $v \in I^N$, the Nucleolus $nu(v)$ is the unique imputation such than $\theta(nu(v)) \leq_L \theta(x)$ for all $x \in I(v)$. The Nucleolus lexicographically minimizes the maximal excess over all possible imputations. With respect to properties we note that the Nucleolus satisfies efficiency, symmetry and the dummy property on I^N. Moreover the Nucleolus is relative invariant with respect to $S-$quivalence. Interestingly we have

Theorem 1.8. *[21] Let $v \in I^N$ be such that $Core(v) \neq \emptyset$. Then $Nu(v) \in Core(v)$.*

One distinctive feature of the Nucleolus is the computational complexity. It is hard to compute the Nucleolus for arbitrary cooperative game but it can be easier in some special case.

Definition 1.6. A cooperative game (N, v) is said to be 1-*convex* if $v(\emptyset) = 0$ and its corresponding *gap function* g^v attains its minimum at the grand coalition N, i.e., for every coalition $S \subseteq N$, $S \neq \emptyset$,

$$0 \leq g^v(N) \leq g^v(S) \qquad \text{where} \qquad g^v(S) = \sum_{i \in S} b_i^v - v(S) \tag{1.3.1}$$

Where $g^v(S) = \sum_{i \in S} b_i^v - v(S)$ and b_i^v is the marginal contribution of player i to coalition N, i.e., $b_i^v = v(N) - v(N \backslash i)$.

For 1-convex games, its Nucleolus agrees with the center of gravity of the Core, of which the extreme points are given by $\vec{b}^v - g^v(N) \cdot \vec{e}_i$, $i \in N$ [21].

1.4 Outline of this thesis

The research that forms the basis of this thesis addresses the following general topics in Cooperative game theory: Why should the players cooperate in cooperative game? Once the coalitions are formed, how to distribute the total value fair and reasonable among all the players? Also, we study the approaches to determine the distribution solution, such as, Shapley value, Nucleolus, Core, Pre-kernel. By solving these problems, we will study the properties of the game model and the value of game.

Chapter 1 contains a short general introduction to the topics of the thesis and gives an overview of the main results, together with some motivation and connections to and relationships with older results.

In Chapter 2, we study the so-called information market game involving n identical firms acquiring a new technology owned by an innovator. For this specific cooperative game, the Nucleolus is determined through a characterization of the symmetrical part of the Core. The non-emptiness of the (symmetrical) Core is shown to be equivalent to one of each, super-additivity, zero-monotonicity, or monotonicity.

In Chapter 3, The Bertrand oligopoly situation with Shubik's demand functions is modeled as a cooperative TU game. For that purpose two optimization programs are solved to arrive at the description of the worth of any coalition in the so-called Bertrand oligopoly game. When the demand's intercept is small, this Bertrand oligopoly game is shown to be a type of cost saving games. Under the complementary circumstances, the Bertrand oligopoly game is shown to be convex and in addition, its Shapley value is fully determined on the basis of linearity applied to an appealing decomposition of the Bertrand oligopoly game into the difference between two convex games, besides one non-essential game.

In Chapter 4, the main goal is to introduce the so-called Service cost savings games involving n different customers requiring service provided by companies. For these specific cooperative games, on one hand, we determine the Shapley value allocation for these service cost savings games through a decomposition method for games into one additive game and one Sharing car pooling cost game, exploiting the linearity of the Shapley value. On the other hand, we determine the Nucleolus allocation as well, by exploiting fully the so-called 1-convexity property for these Service cost savings games.

In Chapter 5, we consider 2-convex n person cooperative TU games. The Nucleolus is determined as some type of constrained equal award rule. Its proof is based on Maschler, Peleg, and Shapley's geometrical characterization for the intersection of the Pre-kernel with the Core. Pairwise bargaining ranges within the Core are required to be in equilibrium. This system of non-linear equations is solved and its unique solution agrees with the Nucleolus.

In Chapter 6, the main goal is twofold. Thanks to the so-called indirect function known as the dual representation of the characteristic function of a coalitional TU game, we derive a new characterization of the Pre-kernel of the coalitional game using the evaluation of its indirect function on the tails of pairwise bargaining ranges arising from a given payoff vector. Secondly, we study two subclasses of coalitional games of which its indirect function has an explicit formula and show the applicability of the determination of the Pre-kernel (Nucleolus) for such types of games using the indirect function. Two such subclasses of games concern the $1-$convex and 2-convex n person games.

In Chapter 7, we illustrate that the so-called indirect function of a cooperative game in characteristic function form is applicable to determine the Nucleolus for a subclass of coalitional games called compromise stable TU games. In accordance with the Fenchel-Moreau theory on conjugate functions, the indirect function is known as the dual representation of the characteristic function of the coalitional game. The key feature of compromise stable TU games is the coincidence of its Core with a box prescribed by certain upper and lower Core bounds. For the purpose of the determination of the Nucleolus, we benefit from the interrelationship between the indirect function and the Pre-kernel of coalitional TU games. The class of compromise stable TU games contains the subclasses of clan games, big boss games, 1- and 2-convex n person TU games. As an adjunct, this chapter reports the indirect function of clan games for the purpose to determine its Nucleolus.

In Chapter 8, we model the interaction between soccer teams and their potential fans as a cooperative cost game based on the annual voluntary sponsorships of fans in order to validate their fan registration in a central database, inspired by the first lustrum of the Club Positioning Matrix (CPM) for professional Dutch soccer teams. The game theoretic approach aims to show that the so-called Nucleolus of the suitably chosen fan data cost game agrees with the deviations of b_i, $i \in N$, from their average, where b_i represents the total budget of sponsorships of fans whose unique favorite soccer team is i.

In chapter 9, The main goal is to reveal the 1-concavity property for a subclass of cost games called Data Cost Games. Two significantly different proofs are treated. The motivation for the study of the 1-concavity property are the appealing theoretical results for both the Core and the Nucleolus, in particular their geometrical characterization as well as their additivity property. The characteristic cost function of the original Data Cost Game assigns to every coalition the additive cost of reproducing the data the coalition does not own. The underlying data and cost sharing situation is composed of three components, namely the player set, the collection of data sets for individuals, and the additive cost function on the whole data set. The first proof of 1-concavity is direct, but robust to a suitable generalization of the characteristic cost function. The second proof of 1-concavity is based on a suitably chosen decomposition of the data cost game which invites to a close comparison between the Nucleolus and the Shapley cost allocations.

In Chapter 10, the topic is two-fold. Firstly, we prove the convexity of Owen's Airport Profit Game (inclusive of revenues and costs). As an adjunct, we characterize the class of 1-convex Airport Profit Games by equivalent properties of the corresponding cost function. Secondly, we classify the class of 1-convex Bankruptcy Games by solving a minimization problem of its corresponding gap function.

Chapter 2

The Core and Nucleolus in a model of information transferal

ABSTRACT - In this chapter, we study the so-called information market game involving n identical firms acquiring a new technology owned by an innovator. For this specific cooperative game, the Nucleolus is determined through a characterization of the symmetrical part of the Core. The non-emptiness of the (symmetrical) Core is shown to be equivalent to one of each, super-additivity, zero-monotonicity, or monotonicity.

2.1 Introduction of the Information market game

Consider the following problem mentioned in [24]. Besides n firms with identical characteristics, there exists an agent called the innovator, having relevant information for the firms. The innovator is not going to use the information for himself, but this information can be sold to the firms. Any firm that decides to acquire the new information (e.g., a new technology) is supposed to make use of the information. The n potential users of the information are the same before and after the innovator offers the new technology. The firms

acquiring the information will be better than before obtaining it, while their utilities are computed under a conservator point of view, assuming that for any uninformed firm, the probability of making the right decision can be described by a binomial probability distribution, being $0 \leq p \leq 1$ the uniform probability of having success. The probability that k among n firms take the right decision is given by $\binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$ and hence, the expected aggregated utility of k firms having success is given by $k \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \cdot u_k$. Here $u_k \geq 0$ represents the utility if k firms make a right decision. Throughout the chapter, the utility function is monotonically decreasing because when the number of firms taking the right decision increases, each firm receives a lower utility level, i.e., $u_{k+1} \leq u_k$ for all $k \geq 1$ (not necessarily normalized in that $u_1 = 1$).
This information trading problem has been modeled by Galdeano et al. as a cooperative game (N, v) in characteristic function form, where the set of firms $N = \{1, 2, \ldots, n+1\}$ consists of the innovator 1, having a new information, and the users $2, 3, \ldots, n+1$, who could be willing to buy the new information. Throughout the thesis, the size (or cardinality) of any coalition $S \subseteq N$ is denoted by s, $0 \leq s \leq n+1$. In case coalition S contains the innovator, then its worth $v(S)$ in the so-called Information market game equals $(s-1) \cdot u_n$ because any member of S, different from the innovator, took the right decision rewarding the expected utility u_n since the $n - s$ uninformed firms outside S are assumed to take right decisions too.

Definition 2.1. [24] The $(n+1)$-person Information market game (N, v) in characteristic function form is given by $v(\emptyset) = 0$, and (cf. Galdeano et al., 2010),

$$v(S) = \begin{cases} (s-1) \cdot u_n, & \text{if} \quad 1 \in S; \\ f_n(s) = \sum\limits_{j=1}^{s} j \cdot \binom{s}{j} \cdot p^j \cdot (1-p)^{s-j} \cdot u_{n-s+j}, & \text{for all} \quad S \neq \emptyset, 1 \notin S. \end{cases} \tag{2.1.1}$$

If the innovator is not a member of coalition S, each one of k successful users rewards an expected utility the amount of $\binom{s}{k} \cdot p^k \cdot (1-p)^{s-k} \cdot u_{n-s+k}$ by assumption of the uninformed users outside S taking the right decisions.

Particularly, the Information market game satisfies $v(\{1\}) = 0$, and $v(\{i\}) = f_n(1) = p \cdot u_n$ for all $i \in N$, $i \neq 1$. Furthermore, $v(N) = n \cdot u_n$, $v(N\backslash\{i\}) = (n-1) \cdot u_n$ for all $i \in N$, $i \neq 1$, whereas $v(N\backslash\{1\}) = f_n(n)$. Consequently, the *marginal contributions* $b_i^v = v(N) - v(N\backslash\{i\})$, $i \in N$, are given by $b_i^v = u_n$ for all $i \in N$, $i \neq 1$, whereas $b_1^v = n \cdot u_n - f_n(n)$. It is left to the reader to verify

$$v(N) - v(S) = \sum_{i \in N\backslash S} \Big[v(N) - v(N\backslash\{i\}) \Big] \qquad \text{for all } S \subseteq N \text{ with } 1 \in S \quad (2.1.2)$$

The case $p = 1$ yields $v(S) = s \cdot u_n$ for all $S \subseteq N\backslash\{1\}$ and so, it concerns the inessential (additive) game corresponding with the vector $(0, u_n, u_n, \ldots, u_n) \in R^{n+1}$. The case $p = 0$ yields zero worth to all coalitions not containing the innovator and so, it concerns the so-called big boss game [46] (with the innovator acting as the big boss). We summarize the main result(s) of Galdeano et al. (2010)

Theorem 2.1. *For the $(n+1)$-person Information market game (N, v) of the form (2.1.1), the following three statements are equivalent.*
(i) Zero-monotonicity, i.e.,

$$v(S \cup \{i\}) \geq v(S) + v(\{i\}) \quad \textit{for all } i \in N \textit{ and all } S \subseteq N\backslash\{i\} \qquad (2.1.3)$$

(ii) $s \cdot u_n \geq f_n(s)$ for all $1 \leq s \leq n$
(iii) (cf. Galdeano et al. , Theorem 2, page 25)

$$\frac{u_n}{u_1} \geq \frac{p \cdot (1-p)^{n-2}}{1 + p \cdot (1-p)^{n-2}} \qquad \textit{applied to the normalization } u_1 = 1 \quad (2.1.4)$$

Besides their study of zero-monotonicity, Galdeano et al. determine the Shapley value of the Information market game [24](Theorem 4, page 27) and compare the Shapley value with the important outcome [24](Theorem 7, page 29) in the non-cooperative model analyzed by [51]. The main goal of the current chapter is to determine the Nucleolus of the Information market game and for that purpose, we explore and characterize the symmetrical part of the Core, provided non-emptiness of the Core.

2.2 Properties of the Information market game

This section reports properties of the characteristic function for the Information market game. In fact, we claim the equivalence of three game properties (called super-additivity, zero-monotonicity, and monotonicity). The proof of their equivalence is based on the monotonically increasing average profit function for coalitions not containing the innovator, i.e., $\frac{f_n(s)}{s} \leq \frac{f_n(s+1)}{s+1}$ for all $1 \leq s \leq n-1$. This significant property has not been discovered before and allows us to report an equivalence theorem which sharpens the previous Theorem2.1.

Definition 2.2. Generally speaking, a cooperative game (N, v) in characteristic function form is said to be *super-additive, zero-monotonic, and monotonic* respectively if its characteristic function v satisfies $v(\emptyset) = 0$ and
(i) $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. (Super-additivity)
(ii) $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ for all $i \in N$ and all $S \subseteq N \backslash \{i\}$. (Zero-monotonicity)
(iii) $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. (Monotonicity)

Theorem 2.2. *For the $(n+1)$-person Information market game (N, v) of the form (2.1.1), the following four statements are equivalent.*
(i)Super-additivity
(ii)Zero-monotonicity
(iii)Monotonicity
(iv)$\frac{f_n(n)}{n} \leq u_n$

Obviously, super-additivity implies zero-monotonicity and in turn, zero-monotonicity implies monotonicity (for non-negative games). The proof of the Equivalence Theorem 2.2 will be based on the fundamental lemma concerning the monotonicity of averaging the profit function $f_n(s)$ of the form (2.1.1).

Lemma 2.1. *The average function given by* $\frac{f_n(s)}{s} = \sum_{j=1}^{s} \binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s-j} \cdot u_{n-s+j}$ *satisfies*
(i) $\frac{f_n(s)}{s} \leq \frac{f_n(s+1)}{s+1}$ *for all* $1 \leq s \leq n-1$.
(ii) $f_n(s+t) \geq f_n(s) + f_n(t)$ *for all* $1 \leq s, t \leq n-1$ *with* $s + t \leq n$.

Proof of Lemma 2.1. Let $1 \leq s \leq n-1$. Concerning the case $s = 1$, note that $f_n(1) = p \cdot u_n$ as well as $f_n(2) = 2 \cdot p \cdot (1-p) \cdot u_{n-1} + 2 \cdot p^2 \cdot u_n$ and so, the inequality $f_n(2) \geq 2 \cdot f_n(1)$ holds due to the fact $(1-p) \cdot u_{n-1} + p \cdot u_n \geq u_n$. Generally speaking, the proof is based on the combinatorial relationship

$\binom{s}{j-1} = \binom{s-1}{j-1} + \binom{s-1}{j-2}$ for all $2 \leq j \leq s$ and proceeds as follows.

$$\begin{aligned}
\frac{f_n(s+1)}{s+1} &= \sum_{j=1}^{s+1} \binom{s}{j-1} \cdot p^j \cdot (1-p)^{s+1-j} \cdot u_{n-s-1+j} = p \cdot (1-p)^s \cdot u_{n-s} \\
&+ p^{s+1} \cdot u_n + \sum_{j=2}^{s} \left[\binom{s-1}{j-1} + \binom{s-1}{j-2} \right] \cdot p^j \cdot (1-p)^{s+1-j} \cdot u_{n-s-1+j} \\
&= p \cdot (1-p)^s \cdot u_{n-s} + \sum_{j=2}^{s} \binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s+1-j} \cdot u_{n-s-1+j} \\
&+ p^{s+1} \cdot u_n + \sum_{j=2}^{s} \binom{s-1}{j-2} \cdot p^j \cdot (1-p)^{s+1-j} \cdot u_{n-s-1+j} \\
&= p \cdot (1-p)^s \cdot u_{n-s} + \sum_{j=2}^{s} \binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s+1-j} \cdot u_{n-s-1+j} \\
&+ p^{s+1} \cdot u_n + \sum_{k=1}^{s-1} \binom{s-1}{k-1} \cdot p^{k+1} \cdot (1-p)^{s-k} \cdot u_{n-s+k} \\
&= \sum_{j=1}^{s} \binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s-j} \cdot \left[(1-p) \cdot u_{n-s-1+j} + p \cdot u_{n-s+j} \right] \\
&\geq \sum_{j=1}^{s} \binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s-j} \cdot u_{n-s+j} = \tfrac{f_n(s)}{s}
\end{aligned}$$

where the relevant inequality holds because the monotonically decreasing sequence $(u_k)_{k \in N}$ satisfies $(1-p) \cdot u_{n-s-1+j} + p \cdot u_{n-s+j} \geq u_{n-s+j}$ for all $1 \leq j \leq s$. This proves part (i).

Concerning part (ii), suppose without loss of generality, $1 \leq s \leq t \leq n-1$ with $s + t \leq n$. By applying part (i) twice, we obtain

$$f_n(s+t) \geq (s+t) \cdot \frac{f_n(t)}{t} = f_n(t) + s \cdot \frac{f_n(t)}{t} \geq f_n(t) + f_n(s)$$

□

Proof of Theorem 2.2. The super-additivity condition for disjoint, non-empty coalitions $S, T \subseteq N \backslash \{1\}$ (not containing the innovator 1) reduces to

$f_n(s+t) \geq f_n(s) + f_n(t)$, which inequality holds by Lemma 2.1(ii). For disjoint, non-empty coalitions $S, T \subseteq N$ with $1 \in T$, $1 \notin S$, it holds that $v(S \cup T) - v(T) = (s+t-1) \cdot u_n - (t-1) \cdot u_n = s \cdot u_n = v(S \cup \{1\})$ and so, the corresponding super-additivity condition reduces to $v(S) \leq v(S \cup \{1\})$ or equivalently, $f_n(s) \leq s \cdot u_n$ for all $1 \leq s \leq n$. By Lemma 2.1(i), it is necessary and sufficient that $\frac{f_n(n)}{n} \leq u_n$. This proves the equivalence Theorem 2.2(i) and Theorem 2.2(iv).
The zero-monotonicity condition for coalitions S containing the innovator are redundant (since $u_n \geq p \cdot u_n$). Among coalitions S not containing the innovator, the zero-monotonicity condition reduces to either $f_n(s+1) \geq f_n(s) + f_n(1)$, which inequality holds by Lemma 2.1(ii), or $s \cdot u_n \geq f_n(s)$. As before, it is necessary and sufficient that $u_n \geq \frac{f_n(n)}{n}$.
Finally, note that the monotonicity condition requires $v(S) \leq v(S \cup \{1\})$ for all $S \subseteq N \backslash \{1\}$, $S \neq \emptyset$, or equivalently, $f_n(s) \leq s \cdot u_n$ for all $1 \leq s \leq n$. □

2.3 The Core of the Information market game

Generally speaking, marginal contributions of players are well-known as upper bounds for pay-offs according to Core allocations, that is $x_i \leq v(N) - v(N \backslash \{i\})$ for all $i \in N$ and all $x \in C(N, v)$. Throughout this chapter, given a pay-off vector $x = (x_i)_{i \in N} \in R^{n+1}$ and a coalition $S \subseteq N$, we denote $x(S) = \sum_{i \in S} x_i$, where $x(\emptyset) = 0$. The Core allocations are selected through *efficiency and group rationality.* The Core, however, is a set-valued solution concept which fails to satisfy the symmetry property in that users of the same type(symmetrical players) receive identical pay-offs according to Core allocations. In order to determine the single-valued solution concept called Nucleolus [53], being some symmetrical Core allocation, our main goal is to investigate the symmetrical part of the Core.

Definition 2.3. The following are the definitions of Core and symmetrical Core for Information market game.
(i) The Core of Information market game is:

$$C(N, v) = \{x \in R^{n+1} \mid x(N) = v(N), x(S) \geq v(S) \quad \text{for all } S \subseteq N\} \quad (2.3.1)$$

(ii) The symmetrical Core allocations require equal pay-offs to users, that is

$$SymCore(N, v) = \{\vec{x} = (x_i)_{i \in N} \in C(N, v) \mid x_2 = x_3 = \ldots = x_{n+1}\} \quad (2.3.2)$$

Lemma 2.2. *(i) Any game (N, v) with a non-empty Core, $C(N, v) \neq \emptyset$, satisfies $v(N) \geq v(N\backslash\{i\}) + v(\{i\})$ for all $i \in N$.*
(ii) In case $p = 1$, the Core of the Information market game is a singleton such that $C(N, v) = \{(0, u_n, u_n, \ldots, u_n)\}$.
(iii) In case $0 \leq p < 1$, if the Information market game possesses a non-empty Core, then $b_1^v \geq 0$, or equivalently, $n \cdot u_n \geq f_n(n)$.
(iv) If $\vec{x} = (x_i)_{i \in N}$ satisfies $\vec{x}(N) = v(N)$ as well as $x_i \leq v(N) - v(N\backslash\{i\})$ for all $i \in N$, $i \neq 1$, then the Core constraints $\vec{x}(S) \geq v(S)$ are redundant for all coalitions $S \subseteq N$ with $1 \in S$.

Proof. (i) Choose $\vec{x} \in C(N, v)$, if Core is non-empty. Clearly, by (2.3.1), for all $i \in N$,

$$v(N) = \vec{x}(N) = \vec{x}(N\backslash\{i\}) + x_i \geq v(N\backslash\{i\}) + x_i \geq v(N\backslash\{i\}) + v(\{i\})$$

(ii) In case $p = 1$, then the Core-constraints $v(\{i\}) \leq x_i \leq v(N) - v(N\backslash\{i\})$ reduce to $p \cdot u_n \leq x_i \leq u_n$ and so, $x_i = u_n$ for all $\vec{x} \in C(N, v)$, and all $i \in N$, $i \neq 1$. Consequently, by efficiency, $x_1 = 0$. The resulting vector $(0, u_n, u_n, \ldots, u_n)$ does indeed satisfy all the Core constraints.
(iii) In case $0 \leq p < 1$, apply part (i) to the Information market game to conclude that $b_1^v = v(N) - v(N\backslash\{1\}) \geq v(\{1\}) = 0$ and so, $b_1^v \geq 0$, or equivalently, $n \cdot u_n \geq f_n(n)$.
(iv) Under the given circumstances, $1 \in S$, together with (2.1.2), we derive the following:

$$\vec{x}(S) = v(N) - \vec{x}(N\backslash S) \geq v(N) - \sum_{i \in N\backslash S} \Big[v(N) - v(N\backslash\{i\}) \Big] = v(S)$$

□

Theorem 2.3. *For the $(n+1)$-person Information market game (N, v) of the form (2.1.1) with $0 \leq p < 1$, the following five statements are equivalent.*
(i) The Core is non-empty, $C(N, v) \neq \emptyset$
(ii) The symmetrical Core is non-empty, $SymCore(N, v) \neq \emptyset$
(iii) $b_1^v \geq 0$
(iv) $\frac{f_n(n)}{n} \leq u_n$
(v) The game fulfills one of the following properties: super-additivity, zero-monotonicity, monotonicity.

Proof. The implication $(i) \Longrightarrow (iii)$ is due to Lemma 2.2(iii). Notice the equivalences $(iii) \Longleftrightarrow (iv)$ as well as $(iv) \Longleftrightarrow (v)$. The implication $(ii) \Longrightarrow (i)$ is trivial. It remains to show the implication $(iv) \Longrightarrow (ii)$, the proof of which will be postponed till Section 2.4(see 2.1(i)). □

Remark 2.1. The condition $\frac{f_n(n)}{n} \leq u_n$ is equivalent to $g_n(p) \leq g_n(1)$ where the function $g_n : [0,1] \to R$ is defined by

$$g_n(p) = p \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{n-1-k} \cdot u_{k+1} \qquad \text{for all } 0 \leq p \leq 1. \quad (2.3.3)$$

Note that p is treated as a variable and that the function satisfies $g_n(1) = u_n$. It is known that any function of the form $g(p) = p^a \cdot (1-p)^b$ is monotonically increasing on the interval $[0, \frac{a}{a+b}]$ and monotonically decreasing on the interval $[\frac{a}{a+b}, 1]$ such that its maximum is attained by $p = \frac{a}{a+b}$ at level $g(\frac{a}{a+b}) = \frac{a^a \cdot b^b}{(a+b)^{a+b}}$. In our framework, the function $g_n(p)$ is composed as the sum of n functions, each of one is monotonically increasing on the subinterval $[0, \frac{k+1}{n}]$ and monotonically decreasing on the sub-interval $[\frac{k+1}{n}, 1]$ such that its maximum value equals $\frac{(k+1)^{k+1} \cdot (n-1-k)^{(n-1-k)}}{n^n}$. On the final interval $[\frac{n-1}{n}, 1]$ all the components are monotonically decreasing, except for the very last component given by $u_n \cdot p^n$. Further investigation about the graph of the function $g_n(p)$ is desirable.

2.4 The Nucleolus of the Information market game

A direct consequence of Lemma 2.2(iv) and Lemma 2.1(i) is the following characterization of the symmetrical part of the Core.

Corollary 2.1. *For the Information market game,*
(i) A symmetrical pay-off vector of the form $\vec{x}(\alpha) = (n \cdot (u_n - \alpha), \alpha, \alpha, \ldots, \alpha) \in R^{n+1}$ *is a Core allocation if and only if* $\alpha \leq u_n$ *and* $s \cdot \alpha \geq f_n(s)$ *for all* $1 \leq s \leq n$*, or equivalently,*

$$\frac{f_n(s)}{s} \leq \alpha \leq u_n \qquad \textit{where} \qquad \frac{f_n(s)}{s} = \sum_{j=1}^{s} \binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s-j} \cdot u_{n-s+j} \quad (2.4.1)$$

(ii) A symmetrical pay-off vector

$$(n \cdot (u_n - \alpha), \alpha, \alpha, \ldots, \alpha) \in SymCore(N, v) \quad \textit{if and only if} \quad \frac{f_n(n)}{n} \leq \alpha \leq u_n$$

(2.4.2)

where $\frac{f_n(n)}{n} = \sum_{j=1}^{n} \binom{n-1}{j-1} \cdot p^j \cdot (1-p)^{n-j} \cdot u_j = p \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{n-1-k} \cdot u_{k+1}$

Definition 2.4. Recall the definitions of excess, Nucleolus and surplus which have been defined in Chapter 1.
(i) Define the *excess* of coalition $S \subseteq N$, $S \neq \emptyset$, at pay-off vector $\vec{x}$ in any cooperative game (N, v) by $e^v(S, \vec{x}) = v(S) - \vec{x}(S)$. Notice that all the excesses of coalitions at Core allocations are non-positive.
(ii) The *excess vector* $\theta(\vec{x}) \in R^{2^n-1}$ at pay-off vector $\vec{x}$ in any n-person game (N, v) has as its coordinates the excesses $e^v(S, \vec{x})$, $S \subseteq N$, $S \neq \emptyset$, arranged in non-increasing order.
(iii) The Nucleolus [53] of a cooperative game (N, v) is the unique pay-off vector $\vec{y}$ of which the excess vector $\theta(\vec{y})$ satisfies the lexicographic order $\theta(\vec{y}) \leq_L \theta(\vec{x})$ for any pay-off vector $\vec{x}$ satisfying efficiency and individual rationality (i.e., $\vec{x}(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$).
(iv) The *surplus* $s^v_{ij}(\vec{x})$ of a player $i \in N$ over another player $j \in N$ at pay-off vector $\vec{x}$ in any cooperative game (N, v) is given by the maximal excess among coalitions containing player i, but not containing player j. That is,

$$s^v_{ij}(\vec{x}) = \max\Big[e^v(S, \vec{x}) \mid S \subseteq N, i \in S, j \notin S\Big] \tag{2.4.3}$$

For the purpose of the determination of the Nucleolus of the Information market game, the next lemma reports about the maximal excess levels at symmetrical pay-off vectors $\vec{x}(\alpha) = (n \cdot (u_n - \alpha), \alpha, \alpha, \ldots, \alpha) \in R^{n+1}$

Lemma 2.3. *For the $(n+1)$-person Information market game (N, v) of the form (2.1.1) it holds:*
(i) $e^v(S, \vec{x}(\alpha)) = -(n+1-s) \cdot (u_n - \alpha)$ *for all $S \subseteq N$ with $1 \in S$. In case $\alpha \leq u_n$, then the maximal excess among nontrivial coalitions containing player 1 equals $\alpha - u_n$ attained at n-person coalitions of the form $N \backslash \{i\}$, $i \neq 1$.*
(ii) $e^v(S, \vec{x}(\alpha)) = f_n(s) - s \cdot \alpha$ *for all $S \subseteq N$, $S \neq \emptyset$, with $1 \notin S$. In case $\frac{f_n(n)}{n} \leq \alpha$, there is no general conclusion about the maximal excess among coalitions not containing player 1.*

Proof (i) For all $S \subseteq N$ with $1 \in S$ it holds

$$\begin{aligned} e^v(S, \vec{x}(\alpha)) &= v(S) - \vec{x}(\alpha)(S) = (s-1) \cdot u_n - \Big[n \cdot u_n - n \cdot \alpha + (s-1) \cdot \alpha \Big] \\ &= -(n+1-s) \cdot (u_n - \alpha) \end{aligned}$$

Under the additional assumption $\alpha \leq u_n$, we obtain $-(n+1-s) \cdot (u_n - \alpha) \leq -(u_n - \alpha)$, that is the maximum is attained for n-person coalitions of the form $N \backslash \{i\}$, $i \neq 1$, (provided $S \neq N$). On the other, for all $S \subseteq N$, $S \neq \emptyset$, with $1 \notin S$, it holds $e^v(S, \vec{x}(\alpha)) = v(S) - \vec{x}(\alpha)(S) = f_n(s) - s \cdot \alpha$. □

Theorem 2.4. *Suppose that the symmetrical Core of the* $(n+1)$*-person Information market game is non-empty, that is* $u_n \geq \frac{f_n(n)}{n}$. *Let* $1 \leq t \leq n$ *be a maximizer in that*

$$\frac{f_n(t) + u_n}{t+1} \geq \frac{f_n(s) + u_n}{s+1} \qquad \textit{for all } 1 \leq s \leq n. \tag{2.4.4}$$

Let $\bar{\alpha} = \frac{f_n(t)+u_n}{t+1}$ *and* $\vec{x}(\bar{\alpha}) = (n \cdot (u_n - \bar{\alpha}), \bar{\alpha}, \bar{\alpha}, \ldots, \bar{\alpha}) \in R^{n+1}$.
(i) Then the pay-off vector $\vec{x}(\bar{\alpha})$ *belongs to the symmetrical Core in that* $\frac{f_n(n)}{n} \leq \bar{\alpha} \leq u_n$.
(ii) The Nucleolus of the $(n+1)$*-person Information market game equals* $\vec{x}(\bar{\alpha})$.

Proof. Suppose $n \cdot u_n \geq f_n(n)$. The following equivalences hold:

$$\bar{\alpha} \leq u_n \quad \text{iff} \quad \frac{f_n(t) + u_n}{t+1} \leq u_n \quad \text{iff} \quad f_n(t) \leq t \cdot u_n \quad \text{iff} \quad \frac{f_n(t)}{t} \leq u_n$$

By Lemma 2.1(i), the latter inequality holds since $\frac{f_n(t)}{t} \leq \frac{f_n(n)}{n} \leq u_n$. So, on the one hand, $\bar{\alpha} \leq u_n$. On the other, from (2.4.4) applied to $s = n$ as well as the assumption $u_n \geq \frac{f_n(n)}{n}$, it follows:

$$\bar{\alpha} = \frac{f_n(t) + u_n}{t+1} \geq \frac{f_n(n) + u_n}{n+1} \geq \frac{f_n(n) + \frac{f_n(n)}{n}}{n+1} = \frac{f_n(n)}{n}$$

(ii) From part (i) and Lemma 2.3(i), on the one hand, we derive the following:

$$\begin{aligned} s^v_{12}(\vec{x}(\bar{\alpha})) &= \max\Big[e^v(S, \vec{x}(\bar{\alpha})) \mid S \subseteq N, 1 \in S, 2 \notin S\Big] \\ &= \max\Big[-(n+1-s)\cdot(u_n - \bar{\alpha}) \mid 1 \leq s \leq n\Big] \\ &= -(u_n - \bar{\alpha}) \qquad \text{and on the other} \\ s^v_{21}(\vec{x}(\bar{\alpha})) &= \max\Big[e^v(S, \vec{x}(\bar{\alpha})) \mid S \subseteq N, 2 \in S, 1 \notin S\Big] \\ &= \max\Big[f_n(s) - s\cdot\bar{\alpha} \mid 1 \leq s \leq n\Big] = \bar{\alpha} - u_n \end{aligned}$$

where the latter equality is due to the choice of $\bar{\alpha}$. The equality $s^v_{12}(\vec{y}) = s^v_{21}(\vec{y})$ for $\vec{y} = \vec{x}(\bar{\alpha})$ suffices to conclude that the Nucleolus is given by $\vec{x}(\bar{\alpha})$. □
Notice that $-s^v_{12}(\vec{x}(\bar{\alpha})) = u_n - \bar{\alpha}$ represents the maximal bargaining range within the Core by transferring money from player 1 to player 2 starting at Core allocation $\vec{x}(\bar{\alpha})$ while remaining in the Core. By Lemma 2.2(iv), recall the redundancy of Core constraints induced by coalitions containing player 1, so no lower bound for Core allocations to player 1.

If the worth of any coalition not containing player 1 is zero (for instance, the big boss games), that is $f_n(s) = 0$ for all $1 \leq s \leq n$, then Theorem 2.4 applies with $t = 1$, $\bar{\alpha} = \frac{u_n}{2}$, yielding the Nucleolus to simplify to $\frac{u_n}{2}\cdot(n, 1, 1, \ldots, 1)$. Thus, the Nucleolus pay-off to the big boss equals the aggregate pay-off to all the users.

Remark 2.2. Concerning the case $t = n$.
Recall that $b^v_1 = n\cdot u_n - f_n(n)$ as well as $b^v_i = u_n$ for all $i \in N$, $i \neq 1$. Thus, the case $t = n$ yields $\bar{\alpha} = \frac{f_n(n)+u_n}{n+1} = u_n - \frac{b^v_1}{n+1} = b^v_i - \frac{b^v_1}{n+1}$ for all $i \in N$, $i \neq 1$. In words, in this setting, the Nucleolus coincides with the center of gravity of $n+1$ vectors given by $\vec{b}^v - \beta\cdot\vec{e}_i$, $i \in N$. Here $\beta = b^v_1$ and $\vec{e}_i$ is the i-th standard vector in R^{n+1}. Note that, for any $1 \leq s \leq n$, the underlying condition $\frac{f_n(n)+u_n}{n+1} \geq \frac{f_n(s)+u_n}{s+1}$ may be rewritten as

$$s\cdot f_n(n) - n\cdot f_n(s) + \Big[f_n(n) - f_n(s)\Big] \geq (n-s)\cdot u_n \tag{2.4.5}$$

Remark 2.3. Inspired by the description of the Nucleolus as given in Remark 2.2, we review a specific subclass of cooperative games with a similar conclusion concerning the Nucleolus.

The $(n+1)$-person Information market game satisfies $b_i^v = u_n$ for all $i \in N$, $i \neq 1$, and so, its gap function g^v is given by $g^v(S) = b_1^v = n \cdot u_n - f_n(n)$ for all $S \subseteq N$ with $1 \in S$ and $g^v(S) = s \cdot u_n - f_n(s)$ otherwise. Consequently, the $(n+1)$-person Information market game of the form (2.1.1) satisfies 1-convexity if and only if any slope $\Delta(f_n)(s) = \frac{f_n(n)-f_n(s)}{n-s}$, $1 \leq s \leq n-1$, is bounded from below by the utility u_n in that $\Delta(f_n)(s) \geq u_n$, together with $\Delta(f_n)(0) \leq u_n$ (provided $f_n(0) = 0$). Observe that the latter condition, together with Lemma 2.1(i), imply the validity of (2.4.5) with reference to the case $t = n$ of Theorem 2.4. To conclude, the 1-convexity property for $(n + 1)$-person Information market games is part of the case $t = n$ and the current procedure for the determination of the Nucleolus agrees with the known approach being the center of gravity of the non-empty Core.

Remark 2.4. A cooperative game (N, v) is said to be 2-*convex* [21] if $v(\emptyset) = 0$ and its corresponding gap function g^v satisfies

$$g^v(N) \leq g^v(S) \qquad \text{for all } S \subseteq N \text{ with } s \geq 2 \text{ and} \tag{2.4.6}$$

$$g^v(\{i\}) \leq g^v(N) \leq g^v(\{i\}) + g^v(\{j\}) \qquad \text{for all } i, j \in N,\ i \neq j \tag{2.4.7}$$

Recall $g^v(N) = g^v(\{1\}) = b_1^v$ and $g^v(\{i\}) = (1-p) \cdot u_n$ for all $i \neq 1$. Together with $b_1^v = n \cdot u_n - f_n(n)$, it follows that (2.4.7) reduces to $(1-p) \cdot u_n \leq b_1^v \leq 2 \cdot (1-p) \cdot u_n$ or equivalently,

$$(n - 2 + 2 \cdot p) \cdot u_n \leq f_n(n) \leq (n - 1 + p) \cdot u_n \tag{2.4.8}$$

Consequently, the $(n+1)$-person Information market game satisfies 2-convexity if and only if (2.4.8) holds as well as any slope $\Delta(f_n)(s)$, $2 \leq s \leq n-1$, is bounded from below by u_n. Particularly, (2.4.5) holds for all $2 \leq s \leq n-1$. Finally, it is left to the reader to derive from (2.4.8) the relevant inequality involving $s = 1$. That is,

$$\frac{f_n(n) + u_n}{n+1} \geq \frac{f_n(1) + u_n}{2} \quad \text{provided } n \geq 3 \text{ and } 0 \leq p < 1, \text{ where } f_n(1) = p \cdot u_n$$

In summary, in the setting of Theorem 2.4, the case $t = n$ applies to $(n + 1)$-person Information market games which are 2-convex. Particularly, the current procedure for the determination of the Nucleolus agrees with the known approach valid for 2-convex games [16].

Next we will give one three person Information market game to show how the Core and Nucleolus of this kind of game look like.

Example 2.1. The three-person Information market game (N, v) (with $n = 2$) is given as follows:

Coalition S	**{1}**	**{2}**	**{3}**	**{1,2}**	**{1,3}**	**{2,3}**	**{1,2,3}**
Worth $v(S)$	0	$p \cdot u_2$	$p \cdot u_2$	u_2	u_2	$f_2(2)$	$2 \cdot u_2$
Gap $g^v(S)$	b_1^v	$(1-p) \cdot u_2$	$(1-p) \cdot u_2$	b_1^v	b_1^v	b_1^v	b_1^v

Note that $b_i^v = u_2$ for $i = 2, 3$, as well as $b_1^v = 2 \cdot u_2 - f_2(2)$, where $f_2(2) = 2 \cdot p \cdot \left[p \cdot u_2 + (1-p) \cdot u_1 \right]$. Here $b_1^v \geq 0$ is a necessary and sufficient condition for non-emptiness of the Core. The three-person Information market game is 1-convex if, besides $b_1^v \geq 0$, one of the following equivalences hold:
(i) $b_1^v \leq (1-p) \cdot u_2$
(ii) $\frac{u_2}{u_1} \leq \frac{2 \cdot p}{2 \cdot p + 1}$
(iii) $p \geq \frac{A}{2}$ where $A = \frac{u_2}{u_1 - u_2}$
Its Core is described by the constraints $x_1 + x_2 + x_3 = 2 \cdot u_2$, and $p \cdot u_2 \leq x_i \leq u_2$ for $i = 2, 3$, as well as $0 \leq x_1 \leq b_1^v$. The constraint $x_1 \geq 0$ is redundant, while the constraint $b_1^v \geq 0$ is a necessary and sufficient condition for non-emptiness of the Core. We distinguish two cases concerning the Core structure, depending on the location of the Core constraint $x_1 = b_1^v$ with respect to the parallel line $x_1 = (1-p) \cdot u_2$. In case $b_1^v \leq (1-p) \cdot u_2$, then the Core is a triangle with three vertices $(0, u_2, u_2)$, $(b_1^v, u_2 - b_1^v, u_2)$ and $(b_1^v, u_2, u_2 - b_1^v)$, representing the Core of a 1-convex three-person game. Its Nucleolus is given by the center of the Core, that is $(b_1^v, u_2, u_2) - \frac{b_1^v}{3} \cdot (1, 1, 1)$.
In case $b_1^v > (1-p) \cdot u_2$, then the Core has five vertices $u_2 \cdot (0, 1, 1)$, $u_2 \cdot (1-p, 1, p)$, $u_2 \cdot (1-p, p, 1)$, $(b_1^v, p \cdot u_2, (2-p) \cdot u_2 - b_1^v)$, and $(b_1^v, (2-p) \cdot u_2 - b_1^v, p \cdot u_2)$ representing the Core of a convex three-person game (with respect to its imputation set).
Concerning the condition (2.4.4), the following equivalences hold (provided $0 \leq p < 1$):
(i) $\frac{f_2(2) + u_2}{3} \geq \frac{f_2(1) + u_2}{2}$
(ii) $\frac{u_2}{u_1} \leq \frac{4 \cdot p}{4 \cdot p + 1}$

(iii) $p \geq \frac{A}{4}$, where $A = \frac{u_2}{u_1 - u_2}$.
According to the main Theorem 2.4, to conclude with, if $p \leq \frac{A}{4}$, then $t = 1$, $\bar{\alpha} = \frac{f_2(1)+u_2}{2} = \frac{u_2}{2} + \frac{p \cdot u_2}{2}$ and hence, the parametric representation of the Nucleolus is given by $(u_2, \frac{u_2}{2}, \frac{u_2}{2}) + \frac{u_2}{2} \cdot (-2 \cdot p, p, p)$.
If $p \geq \frac{A}{4}$, then $t = 2$, $\bar{\alpha} = \frac{f_2(2)+u_2}{3} = u_2 - \frac{b_1^v}{3}$ and hence, the parametric representation of the Nucleolus is given by $(0, u_2, u_2) - \frac{1}{3} \cdot (-2 \cdot b_1^v, b_1^v, b_1^v)$.
If p varies upwards from zero till $\frac{A}{4}$, then the Nucleolus starts at $(u_2, \frac{u_2}{2}, \frac{u_2}{2})$ and moves with a speed scaled by $\frac{u_2}{2}$. If p varies downwards from 1 till $\frac{A}{4}$, then the Nucleolus starts at $(0, u_2, u_2)$ and moves with a speed scaled by $b_1^v = 2 \cdot (1-p) \cdot \left[(1+p) \cdot u_2 - p \cdot u_1 \right]$. Anyhow, the Nucleolus moves by two different speeds from $(0, u_2, u_2)$ being the full Core if $p = 1$ till $(u_2, \frac{u_2}{2}, \frac{u_2}{2})$, being the center of the Core if $p = 0$ with four vertices $(2 \cdot u_2, 0, 0)$, $(u_2, u_2, 0)$,$(u_2, 0, u_2)$, and $(0, u_2, u_2)$.

2.5 The Shapley value of the Information market game

Next, we will give an simply approach to determine the Shaply value of the Information market game, comparing the approach given in [24].

Theorem 2.5. *The Shapley value $Sh_1(N, v)$ of the innovator in the $(n+1)$-person Information market game (N, v) equals the difference between one half of the aggregate pay-off and the average worth of coalitions not containing the innovator, that is*

$$Sh_1(N, v) = \frac{n \cdot u_n}{2} - \frac{1}{n+1} \sum_{s=0}^{n} f_n(s) \quad \text{and for all } i \neq 1, \tag{2.5.1}$$

$$Sh_i(N, v) = \frac{u_n}{2} + \frac{1}{n \cdot (n+1)} \cdot \sum_{s=0}^{n} f_n(s) \tag{2.5.2}$$

Proof. Put $f_n(0) = 0$. Using its classical formula [56], the Shapley value of the innovator 1 is determined as following:

$$\begin{aligned}
Sh_1(N,v) &= \sum_{S \subseteq N\setminus\{1\}} \frac{s!\cdot(n-s)!}{(n+1)!}\cdot\Big[v(S\cup\{1\}) - v(S)\Big] \\
&= \sum_{S \subseteq N\setminus\{1\}} \frac{s!\cdot(n-s)!}{(n+1)!}\cdot v(S\cup\{1\}) - \sum_{S \subseteq N\setminus\{1\}} \frac{s!\cdot(n-s)!}{(n+1)!}\cdot v(S) \\
&= \sum_{S \subseteq N\setminus\{1\}} \frac{s!\cdot(n-s)!}{(n+1)!}\cdot s\cdot u_n - \sum_{S \subseteq N\setminus\{1\}} \frac{s!\cdot(n-s)!}{(n+1)!}\cdot f_n(s) \\
&= \sum_{s=0}^{n} \binom{n}{s}\cdot\tfrac{s!\cdot(n-s)!}{(n+1)!}\cdot s\cdot u_n - \sum_{s=0}^{n} \binom{n}{s}\cdot\tfrac{s!\cdot(n-s)!}{(n+1)!}\cdot f_n(s) \\
&= \sum_{s=0}^{n} \frac{s}{n+1}\cdot u_n - \sum_{s=0}^{n} \frac{f_n(s)}{n+1} = \frac{n\cdot u_n}{2} - \frac{1}{n+1}\cdot\sum_{s=0}^{n} f_n(s)
\end{aligned}$$

□

Remark 2.5. The Shapley value $Sh(N,v)$ is a symmetric allocation which verifies the upper Core bound u_n.

Indeed, by Lemma 2.2(i), it holds $\frac{f_n(n)}{n} \geq \frac{f_n(s)}{s}$ for all $1 \leq s \leq n$ and so,

$$\frac{1}{n\cdot(n+1)}\cdot\sum_{s=0}^{n} f_n(s) \leq \frac{1}{n\cdot(n+1)}\cdot\frac{f_n(n)}{n}\cdot\sum_{s=0}^{n} s = \frac{f_n(n)}{2\cdot n} \leq \frac{u_n}{2}$$

where the last inequality is due to the assumption $f_n(n) \leq n\cdot u_n$. Thus, $Sh_i(N,v) \leq u_n$ for all $i \in N$, $i \neq 1$, whereas the Shapley value for users does not necessarily meet the lower Core bound $\frac{f_n(n)}{n}$. For instance, for the three-person Information market game (with $n = 2$ and $0 \leq p < 1$), the following equivalences hold:

$$Sh_2(N,v) \geq \frac{f_2(2)}{2} \iff \frac{u_2}{u_1} \geq \frac{4\cdot p}{4\cdot p+3} \iff p \leq \frac{3}{4}\cdot A$$

where $A = \frac{u_2}{u_1 - u_2}$. By the super-additivity (or zero-monotonicity) of the Information market game, its Shapley value satisfies individual rationality, that is

$Sh_i(N, v) \geq v(\{i\})$ for all $i \in N$. To conclude, the Shapley value of the Information market game is an imputation, but not necessarily a Core allocation (in spite of the validity of the upper Core bound for users).

Chapter 3

Convexity and the Shapley value in Bertrand Oligopoly TU-games with Shubik's Demand Functions

ABSTRACT - In this chapter, The Bertrand oligopoly situation with Shubik's demand functions is modeled as a cooperative TU game. For that purpose two optimization programs are solved to arrive at the description of the worth of any coalition in the so-called Bertrand oligopoly game. When the demand's intercept is small, this Bertrand oligopoly game is shown to be a type of cost saving games. Under the complementary circumstances, the Bertrand oligopoly game is shown to be convex and in addition, its Shapley value is fully determined on the basis of linearity applied to an appealing decomposition of the Bertrand oligopoly game into the difference between two convex games, besides one non-essential game.

3.1 Introduction

A central problem in oligopoly theory is the existence of collusive behaviors between firms, that is, situations in which firms are able to coordinate and to stabilize their strategies in order to increase their profits. The classical Cournot and Bertrand Oligopoly situations are such examples where firms are better off through cooperation rather than by acting independently. A cartel operating successfully is the OPEC (Organization of the Petroleum Exporting Countries) cartel which restricts oil supply in order to control the oil price market. Another example of a cartel which had operated is the agreement between multinational firms Saint-Gobain, Pilkington, Asahi and Soliver in the flat glass industry. Their illegal agreement on the price of glass in the car industry from 1998 to 2003 had been fined by the European Commission in 2008.
Non-cooperative game theory has provided the theoretical bases for the existence of collusive behaviors between firms by means of repeated games. Under this first approach, each firm does not have any interest in defecting from the collusive behavior because it rationally anticipates future punishments in the periods following its defection. An alternative way to formalize the existence of collusive behaviors comes from Cooperative game theory. Under this second approach, firms are allowed to sign binding agreements in order to form cartels called coalitions. With such an assumption cooperative games called oligopoly TU (Transferable Utility)-games can be defined and the existence of collusive behaviors is then related to the non-emptiness of the Core of such games. Aumann(1959) [2] proposes two different models in order to define the cooperative game: according to the first, every cartel computes the total profit which it can guarantee itself regardless of what outsiders do; the second computes the minimal profit for which outsiders can prevent the firms in the cartel from getting more. These two models lead to consider the α and β-characteristic function forms respectively.

In this chapter, we follow this cooperative approach to analyze collusive behaviors and we study a subclass of oligopoly TU-games in α and β-characteristic function forms. Many works have studied the Core of oligopoly TU-games. As regards Cournot oligopoly TU-games with or without transferable technologies, Zhao (1999) [58] [59] shows that the α and β-characteristic function forms lead to the same class of Cournot oligopoly TU-games. When technologies are transferable, Zhao(1999) [58] provides a necessary and sufficient condition to establish the convexity property in case the inverse demand function and cost functions are linear. Although oligopoly games may fail to be convex in general, Norde et al.(2002) [47] show they are nevertheless totally balanced. When technologies are not transferable, Zhao(1999) [59] proves that the Core of such oligopoly games is non-empty if every individual profit function is continuous and concave. Furthermore, Norde et al.(2002) [47] show that these games are convex in case the inverse demand function and cost functions are linear, and Driessen and Meinhardt(2005,2010) [19] [16] provide economically meaningful sufficient conditions to guarantee the convexity property in a more general case.

As regards Bertrand oligopoly TU-games, Deneckere and Davidson (1985) [9] consider a Bertrand oligopoly situation with differentiated products in which the demand system is Shubik's (1980) [57] and firms operate at a constant and identical marginal and average cost. They prove that these games have a superadditive property in the sense that a merger of two disjoint cartels results in a joint after-merger profit for them which is greater than the sum of their pre-merger profits. Lardon(2010) [36] extends this result by considering the α and β-characteristic functions of these games. As for Cournot oligopoly TU-games, he shows that the α and β-characteristic function forms lead to the same class of Bertrand oligopoly TU-games and proves that the convexity property holds for this class of games.

In this chapter, we deal with the study of Bertrand oligopoly TU-games in α and β-characteristic function forms with Shubik's demand functions in which firms have possibly distinct marginal costs for all firms. In section 3.2 two subsequent optimization programs are solved to determine the worth of any coalition according to the characteristic function of the cooperative Bertrand Oligopoly TU-game. As a corollary, it is shown in the introductory Section 3.2 that the α and β approach lead to the same class of Bertrand Oligopoly-TU games. This first result should be interpreted as an extension of Lardon's result [36] which applied only for the case with identical marginal costs. Particularly, we distinguish two types of coalitions in Bertrand oligopoly TU-games. On the one hand, if the intercept of demand is sufficiently small, then the Bertrand oligopoly TU-games share clear similarities with a type of cost savings game.

Section 3.3 states, if the intercept of demand is sufficiently large, then Bertrand oligopoly TU-games are convex. In addition, in Section 3.4, the well-known game theoretic solution called Shapley value is fully determined on the basis of linearity applied to a decomposition of Bertrand oligopoly TU- games into the difference between two convex games, besides one non-essential game. Concluding remarks are presented in the Section 3.5.

3.2 The non-symmetric Bertrand Oligopoly TU Game with Shubik's Demand Functions

In [57] *Bertrand oligopoly situation* is described by a 3-tuple $(N, (D_i)_{i\in N}, (C_i)_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is the finite *set of firms*, such that, for every firm $i \in N$, the *Shubik's demand function* $D_i : R^n_+ \to R$ and the *(linear) cost function* $C_i : R_+ \to R_+$ with marginal cost $c_i \geq 0$ respectively are given by

$$D_i(p_1, p_2, \ldots, p_n) = V - p_i - r \cdot \left[p_i - \frac{1}{n} \cdot \sum_{k\in N} p_k\right] \tag{3.2.1}$$

and $C_i(x) = c_i \cdot x$ for all $x \in R_+$ where p_i is the *price* charged by firm i, the *demand's intercept* $V \geq 0$ when all prices are zero, and let $r > 0$ be the *substitutability parameter*. When r approaches zero, products become unrelated, and when r approaches infinity, products become perfect substitutes. The quantity demanded of firm i's brand depends on its own price p_i and the difference between its own price and the average price in the industry. The latter quantity is decreasing with respect to its own price p_i and increasing with respect to any price p_j, $j \neq i$. Notice that firms may operate at possibly different marginal costs $c_i \geq 0$, $i \in N$, and these marginal costs do not limit the non-negative prices $p_i \geq 0$, $i \in N$, of firms. The corresponding *Bertrand oligopoly game* in normal form $(N, (X_i)_{i\in N}, (\pi_i)_{i\in N})$ is given by player i's *strategy set* $X_i = R_+ = [0, \infty)$ and *individual profit function* $\pi_i : \Pi_{k\in N} X_k \to R$ such that

$$\pi_i(p_1, p_2, \ldots, p_n) = (p_i - c_i) \cdot D_i(p_1, p_2, \ldots, p_n)$$

So, for all $i \in N$,

$$\pi_i(p_1, p_2, \ldots, p_n) = (p_i - c_i) \cdot \left[V - (1+r) \cdot p_i + \tfrac{r}{n} \cdot \sum_{k \in N} p_k\right] \quad (3.2.2)$$

Denote for any $T \subseteq N$, $T \neq \emptyset$, the coalitional strategy set $X_T = \Pi_{k \in T} X_k$ and define the *coalitional profit function* $\pi_T : X_T \times X_{N \setminus T} \to R$ by $\pi_T(p_T, p_{N \setminus T}) = \sum_{k \in T} \pi_k(p_T, p_{N \setminus T})$, for all $(p_T, p_{N \setminus T}) \in X_T \times X_{N \setminus T}$. The corresponding *Bertrand oligopoly game in* α*- and* β*-characteristic function form* (N, v_α) and (N, v_β) are defined, for every coalition $S \subseteq N$, $S \neq \emptyset$, as follows:

$$v_\alpha(S) = \max_{p_S \in X_S} \min_{p_{N \setminus S} \in X_{N \setminus S}} \pi_S(p_S, p_{N \setminus S}) \quad (3.2.3)$$

$$v_\beta(S) = \min_{p_{N \setminus S} \in X_{N \setminus S}} \max_{p_S \in X_S} \pi_S(p_S, p_{N \setminus S}) \quad (3.2.4)$$

$$\pi_S(p_S, p_{N \setminus S}) = \sum_{j \in S} (p_j - c_j) \cdot \left[V - (1+r) \cdot p_j + \tfrac{r}{n} \cdot \sum_{k \in N} p_k\right] \quad (3.2.5)$$

Generally speaking, it is known that the game in β-characteristic function form majorizes the α-characteristic function form, i.e., $v_\beta(S) \geq v_\alpha(S)$ for all $S \subseteq N$, and particularly, both games do not differ for the grand coalition N, i.e., $v_\beta(N) = v_\alpha(N)$ because it concerns only the maximization program (instead of two subsequent optimization programs). As a matter of fact, it turns out that according to the (unique) optimal solution of this maximization problem in the Bertrand oligopoly situation, each firm $i \in N$ charges the price amounting the average of its marginal cost c_i and the demand's intercept V, and so, the coalitional profit function $\pi_N((p_i)_{i \in N})$ attains its maximum whenever $p_i = \frac{V + c_i}{2}$ for all firms $i \in N$, its profits amounting as follows (see Theorem 3.2):

$$v_\beta(N) = \pi_N((\tfrac{V+c_i}{2})_{i \in N}) = \tfrac{n}{4} \cdot \left[V - \bar{c}_N\right]^2 + \tfrac{1+r}{4} \cdot \left[\sum_{j \in N} (c_j)^2 - n \cdot (\bar{c}_N)^2\right] \quad (3.2.6)$$

Through the remainder, for any $S \subseteq N$, $S \neq \emptyset$, let $\bar{c}_S = \frac{1}{s} \cdot \sum_{j \in S} c_j$ denote the average cost of S. The production technology of any firm i is influenced

by its marginal cost c_i in the sense that firms with a small (large respectively) marginal cost possess a high (low respectively) production technology. Because firms are confronted with very different marginal costs, there exist significant productivity gains inside any coalition such that any firm in the coalition S with a low production technology benefits from the high production technology of other firms in S. In this setting, it is natural to replace the marginal costs of individual firms in the non-cooperative framework by the average cost of the coalition in the cooperative framework. Henceforth, the expression $\sum_{j\in N}(c_j)^2 - n\cdot(\bar{c}_N)^2$ represents the (non-negative) cost savings of the grand coalition N, taking into account that the quadratic utility function $u(x) = x^2$ applies. Besides this type of cost savings, the quadratic utility form of the netto demand $V - \bar{c}_N$ is added to determine the worth $v_\beta(N)$ of the grand coalition N.
For strict subcoalitions $S \subseteq N$, $S \neq N$, its worth $v_\beta(S)$ is composed of two contributions, namely a similar type of cost savings, and a critical type of netto demand, under the restriction that the firms in S charge the only non-zero, but common price p, amounting $D_i((p)_{j\in S}, (0)_{j\in N\setminus S}) = V - p\cdot h(s)$. Particularly, if the common price equals the average cost of coalition S, then the critical netto demand equals $V - \bar{c}_S\cdot h(s)$, of which the quadratic utility applies only if the critical netto demand itself is non-negative. Here the decreasing linear function $h: R \to R$ with slope $\frac{-r}{n}$ is defined by $h(x) = 1 + r - \frac{r}{n}\cdot x$ for all $x \in R$, satisfying $h(0) = 1 + r$ and $h(n) = 1$.
The mathematical description of the Bertrand Oligopoly game is presented in the next Theorem 3.1. Its proof is rather lengthy containing four parts and so, it will be postponed till the Theorem 3.2. Further, interpretations of the solutions of the two subsequent optimization programs are given in Theorem 3.2. Moreover, the very last part of the proof ends with the alternative but equivalent formula of the form (3.2.14) for the worth $v_\beta(S)$ of coalition S in the Bertrand oligopoly game.

Theorem 3.1. *Let $S \subseteq N$, $S \neq N$, $S \neq \emptyset$. Whether or not the critical demand $V - h(s)\cdot\bar{c}_S$ is feasible, the worth $v_\beta(S)$ of coalition S in the Bertrand oligopoly game is as following:*

$$v_\beta(S) = \tfrac{s}{4\cdot h(s)}\cdot\Big[V - h(s)\cdot\bar{c}_S\Big]^2 + \tfrac{1+r}{4}\cdot\Big[\sum_{j\in S}(c_j)^2 - s\cdot(\bar{c}_S)^2\Big] \qquad \textit{if } V > h(s)\cdot\bar{c}_S; \tag{3.2.7}$$

$$v_\beta(S) = \tfrac{1+r}{4} \cdot \left[\sum_{j \in S} (c_j)^2 - s \cdot (\bar{c}_S)^2 \right] \qquad \text{if } V \le h(s) \cdot \bar{c}_S. \tag{3.2.8}$$

Corollary 3.1. *Consider the symmetric Bertrand oligopoly situation with common marginal costs, i.e., $c_i = c > 0$ for all $i \in N$. Then the game (N, v_β) in β-characteristic function form satisfies $v_\beta(N) = \frac{n}{4} \cdot (V - c)^2$ and for all $S \subseteq N$, $S \neq N$, $S \neq \emptyset$, the following holds.*
(i) $v_\beta(S) = 0$ whenever the demand's intercept V is small enough, i.e., if $V \le c \cdot h(s)$.
(ii) If the demand's intercept V is large enough, i.e., if $V > c \cdot h(s)$, then

$$v_\beta(S) = \tfrac{s}{4 \cdot h(s)} \cdot \Big[V - c \cdot h(s) \Big]^2 = \tfrac{s}{h(s)} \cdot (\tfrac{c \cdot r}{2 \cdot n})^2 \cdot \Big[E - (n - s) \Big]^2 \qquad \text{provided } E > n - s,$$

where the proportionally aggregate netto demand *E is defined by $E = \frac{n \cdot (V - c)}{c \cdot r}$.*

The non-zero coalitional worth in the symmetric Bertrand oligopoly Game depends on the validity of the constraint $V > c \cdot h(s)$ involving the demand's intercept V or the equivalent constraint $E > n - s$ involving the proportionally aggregate netto demand E. In this setting, we interpret $n \cdot (V - c)$ as the aggregate netto demand when prices are zero.
Obviously, if a coalition S of size s meets the constraint $E \le n - s$ yielding zero worth $v_\beta(S) = 0$, then any coalition of the same size s or less inherits the same constraint yielding zero worth. Similarly, if a coalition T of size t meets the inverse constraint $E > n - t$ yielding non-zero worth $v_\beta(T) > 0$, then any coalition of the same size or more inherits the same inverse constraint yielding non-zero worth. In case the demand's intercept V is large enough, then the coalitional worth in the non-symmetric Bertrand oligopoly Game counts, besides the cost savings, the non-zero worth in the symmetric Bertrand oligopoly game, with the understanding that the constant marginal cost is to be replaced by the average coalitional cost.
According to the very last formula, the per-capita worth $\frac{v_\beta(S)}{s}$ is strategically equivalent to the quotient of the square of a bankruptcy game (with estate E and unitary claims) and a linearly decreasing symmetric game (varying from levels $1 + r$ down to level 1).

Corollary 3.2. *The α- and β-characteristic function forms (N, v_α) and (N, v_β) coincide, that is $v_\alpha(S) = v_\beta(S)$ for all $S \subseteq N$.*

Proof. Fix the coalition $S \subsetneqq N$, $S \neq \emptyset$. It remains to prove the inequality $v_\alpha(S) \geq v_\beta(S)$. We claim the following chain of (in)equalities:

$$\begin{aligned} v_\alpha(S) &= \max_{p_S \in X_S} \min_{p_{N\setminus S} \in X_{N\setminus S}} \pi_S(p_S, p_{N\setminus S}) \\ &\geq \min_{p_{N\setminus S} \in X_{N\setminus S}} \pi_S((\bar{p}_j^S)_{j\in S}, p_{N\setminus S}) \\ &= \pi_S((\bar{p}_j^S)_{j\in S}, \bar{p}_{N\setminus S}) = v_\beta(S) \end{aligned}$$

As shown in Theorem 3.2, the last equality holds because of the construction of both vectors $(\bar{p}_j^S)_{j\in S}$ and $\bar{p}_{N\setminus S}$. The last equality but one holds because $\bar{p}_{N\setminus S}$ is a minimizer of the minimization program $\min_{p_{N\setminus S} \in X_{N\setminus S}} \pi_S((\bar{p}_j^S)_{j\in S}, p_{N\setminus S})$. □

Theorem 3.2. *Let $S \subseteq N$, $S \neq N$, $S \neq \emptyset$.*
Solving the minimization program for $N\setminus S$ at the second stage causes the critical demand $V - h(s)\cdot \bar{c}_S$, where $\bar{c}_S$ is the average cost of coalition S and the linearly decreasing function defined by $h(x) = 1 + r - \frac{r\cdot x}{n}$ for all $x \in R$.
(i) If the critical demand is not feasible (i.e., if the demand's intercept V is small enough), then according to the (unique) solution of the maximization program at the first stage, each firm $i \in S$ charges the price amounting the midpoint of its marginal cost c_i and the average cost $\bar{c}_S$, and the coalitional worth $v_\beta(S)$ equals the cost savings of the form (3.2.8). Firms outside S charge in total a multiple of the amount of the critical demand.
(ii) If the critical demand is feasible (i.e., if the demand's intercept V is large enough), then according to the (unique) solution of the maximization program at the first stage, each firm $i \in S$ charges the price amounting the midpoint of its marginal cost c_i and $\frac{V}{h(s)}$ which is strictly larger than the average cost $\bar{c}_S$, and the coalitional worth $v_\beta(S)$ equals the sum of the cost savings and the square of the critical demand of the form (3.2.8). Firms outside S do not charge any price.

Proof. Let $S \subseteq N$, $S \neq \emptyset$.
Part 1. Let $i \in S$. The partial derivative $\frac{\partial \pi_S}{\partial p_i}$ of the coalitional profit function

π_S of the form (3.2.5) is as follows:

$$\begin{aligned}
&\frac{\partial \pi_S}{\partial p_i}(p_S, p_{N\setminus S}) \\
= &\left[V - (1+r)\cdot p_i + \tfrac{r}{n}\sum_{k\in N} p_k\right] + (p_i - c_i)\left[-(1+r) + \tfrac{r}{n}\right] + \sum_{j\in S\setminus\{i\}} (p_j - c_j)\tfrac{r}{n} \\
= &\ V - (1+r)\cdot(2\cdot p_i - c_i) + \tfrac{r}{n}\cdot \sum_{j\in S}(p_j - c_j) + \tfrac{r}{n}\cdot\sum_{k\in N} p_k
\end{aligned}$$

Consequently, the solution to the first order conditions $\frac{\partial \pi_S}{\partial p_i} = 0, i \in S$, satisfies,

$$p_i = \tfrac{c_i}{2} + \tfrac{1}{2\cdot(1+r)}\cdot\left[V + \tfrac{r}{n}\cdot\left[\sum_{j\in S}(p_j - c_j) + \sum_{k\in N} p_k\right]\right] \quad \text{for all} \quad i \in S$$

So far, we conclude that for the solution to the first order conditions (associated with the maximization program) it holds that $p_i - \frac{c_i}{2}$ is constant for all $i \in S$, say $p_i - \frac{c_i}{2} = \frac{\bar{p}_S}{2}$ for every $i \in S$. Through substitution in the latter expression, we arrive at the following relationships:

$$\begin{aligned}
(1+r)\cdot\bar{p}_S &= V + \frac{r}{n}\cdot\left[\sum_{k\in N\setminus S} p_k + \sum_{j\in S}(2\cdot p_j - c_j)\right] \\
&= V + \frac{r}{n}\cdot\left[\sum_{k\in N\setminus S} p_k + s\cdot\bar{p}_S\right] \qquad (3.2.9)
\end{aligned}$$

Rewriting the latter equality yields

$$\begin{aligned}
\left[1 + r - \frac{r\cdot s}{n}\right]\cdot\bar{p}_S &= V + \frac{r}{n}\cdot\sum_{k\in N\setminus S} p_k \quad \text{and thus,} \\
\bar{p}_S &= h(s)^{-1}\cdot\left[V + \frac{r}{n}\cdot\sum_{k\in N\setminus S} p_k\right] \qquad (3.2.10)
\end{aligned}$$

Recall that the solution $\bar{p}_i^S$ to the first order conditions is given by the midpoint of the marginal cost of firm i and the common price charge $\bar{p}_S$ by members of coalition S, i.e., $\bar{p}_i^S = \frac{\bar{p}_S + c_i}{2}$ for all $i \in S$. In case $S = N$, then (3.2.10) reduces to $\bar{p}_N = V$ and so, according to the (unique) optimal solution of this maximization program, each firm charges the price amounting the midpoint of the demand's intercept V and the marginal cost of the firm (as stated at the beginning of Section 3.2).

Part 2. Let $S \neq N$, concerning the subsequent minimization program $\min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S((\bar{p}^S_j)_{j\in S}, p_{N\setminus S})$, we derive from (3.2.5) that its objective function reduces as follows:

$$\begin{aligned}
&\pi_S((\bar{p}^S_j)_{j\in S}, p_{N\setminus S})\\
= &\sum_{j\in S}(\bar{p}^S_j - c_j)\cdot\Big[V-(1+r)\cdot\bar{p}^S_j+\frac{r}{n}\cdot\sum_{k\in S}\bar{p}^S_k+\frac{r}{n}\cdot\sum_{k\in N\setminus S}p_k\Big]\\
= &\sum_{j\in S}\tfrac{\bar{p}_S-c_j}{2}\cdot\Big[V-(1+r)\cdot\tfrac{\bar{p}_S+c_j}{2}+\tfrac{r}{2\cdot n}\cdot[s\cdot\bar{p}_S+s\cdot\bar{c}_S]+\tfrac{r}{n}\cdot\textstyle\sum_{k\in N\setminus S}p_k\Big]\\
= &\sum_{j\in S}\tfrac{\bar{p}_S-c_j}{2}\cdot\Big[V-\tfrac{h(s)}{2}\cdot\bar{p}_S-\tfrac{1+r}{2}\cdot c_j+\tfrac{r\cdot s}{2\cdot n}\cdot\bar{c}_S+\tfrac{r}{n}\cdot\textstyle\sum_{k\in N\setminus S}p_k\Big] \quad (3.2.11)
\end{aligned}$$

By (3.2.10), the partial derivatives of $\bar{p}_S$ with respect to non-members of S do not differ in that $\frac{\partial\bar{p}_S}{\partial p_\ell}=\frac{r}{n\cdot h(s)}$ for all $\ell\in N\backslash S$. Hence, by differentiating (3.2.11), we obtain the following: for all $l\in N\setminus S$,

$$\begin{aligned}
&\frac{\partial\pi_S}{\partial p_\ell}((\bar{p}^S_j)_{j\in S}, p_{N\setminus S})\\
= &\sum_{j\in S}\frac{r}{2\cdot n\cdot h(s)}\cdot\Big[V-\frac{h(s)}{2}\cdot\bar{p}_S-\frac{1+r}{2}\cdot c_j+\frac{r\cdot s}{2\cdot n}\cdot\bar{c}_S+\frac{r}{n}\cdot\sum_{k\in N\setminus S}p_k\Big]\\
+ &\sum_{j\in S}\tfrac{\bar{p}_S-c_j}{2}\cdot\Big[-\tfrac{h(s)}{2}\cdot\tfrac{r}{n\cdot h(s)}+\tfrac{r}{n}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{2 \cdot n \cdot h(s)} \left[sV - \frac{h(s)}{2} \cdot s \cdot \bar{p}_S - \frac{1+r}{2} \cdot s \cdot \bar{c}_S + \frac{r \cdot s^2}{2 \cdot n} \cdot \bar{c}_S + \frac{rs}{n} \cdot \sum_{k \in N \setminus S} p_k \right] \\
&+ \frac{1}{2} \cdot \left[s \cdot \bar{p}_S - s \cdot \bar{c}_S \right] \cdot \frac{r}{2 \cdot n} \\
&= \frac{r \cdot s}{2 \cdot n \cdot h(s)} \cdot \left[V + \frac{r}{n} \cdot \sum_{k \in N \setminus S} p_k \right] - \frac{r \cdot s}{2 \cdot n} \cdot \bar{c}_S
\end{aligned}
$$

Concerning the solution to the minimization program, we conclude that the first order conditions $\frac{\partial \pi_S}{\partial p_\ell}((\bar{p}_j^S)_{j \in S}, p_{N \setminus S}) = 0$ for all $\ell \in N \setminus S$ yield

$$
V + \frac{r}{n} \cdot \sum_{k \in N \setminus S} p_k = h(s) \cdot \bar{c}_S \tag{3.2.12}
$$

Finally, take care about the non-negativity constraint for prices.

Part 3. Suppose the demand's intercept V is small enough in that $V < h(s) \cdot \bar{c}_S$. Under these circumstances, we derive from (3.2.12) and in turn (3.2.10) that it holds

$$
\frac{r}{n} \cdot \sum_{k \in N \setminus S} \bar{p}_k = h(s) \cdot \bar{c}_S - V \quad \text{and} \quad \bar{p}_S = \bar{c}_S \quad \text{and} \quad \bar{p}_i^S = \frac{\bar{c}_S + c_i}{2} \quad \text{for all } i \in S.
$$

In words, the optimal solution is given by the midpoint of the marginal cost of any firm and the average cost of the coalition S. It is left to the reader to check the following relationship:

$$
V - (1+r) \cdot \bar{p}_j^S + \tfrac{r}{n} \cdot \sum_{k \in N} \bar{p}_k = \tfrac{1+r}{2} \cdot [\bar{c}_S - c_j] \qquad \text{for all } j \in S.
$$

Consequently, by (3.2.5), the coalitional profit reduces as follows:

$$\begin{aligned} v_\beta(S) &= \pi_S((\bar{p}_j^S)_{j\in S}, \bar{p}_{N\setminus S}) \\ &= \sum_{j\in S}[\bar{p}_j^S - c_j] \cdot \left[V - (1+r)\cdot \bar{p}_j^S + \frac{r}{n}\cdot \sum_{k\in N}\bar{p}_k\right] \\ &= \frac{(1+r)}{2}\cdot \sum_{j\in S}[\bar{p}_j^S - c_j]\cdot[\bar{c}_S - c_j] = \frac{(1+r)}{4}\cdot \sum_{j\in S}[\bar{c}_S - c_j]\cdot[\bar{c}_S - c_j] \\ &= \frac{(1+r)}{4}\cdot \sum_{j\in S}[\bar{c}_S - c_j]^2 = \frac{(1+r)}{4}\cdot\left[\sum_{j\in S}(c_j)^2 - s\cdot(\bar{c}_S)^2\right] \quad (3.2.13) \end{aligned}$$

This game of the form (3.2.13) has the same similarity with the notion of variance $Var[x] = \delta[(x-\delta[x])^2] = \delta[x^2] - (\delta[x])^2$, thus we call this game variance game.

Part 4. Suppose the demand's intercept V is large enough in that $V \geq h(s)\cdot \bar{c}_S$. Under these circumstances, we derive from (3.2.12) and in turn (3.2.10) that it holds

$$\sum_{k\in N\setminus S}\bar{p}_k = 0 \quad \text{and} \quad \bar{p}_S = \frac{V}{h(s)}$$

Recall that $\bar{p}_i^S = \frac{\bar{p}_S + c_i}{2}$ for all $i \in S$. It is left to the reader to check the following relationship:

$$V - (1+r)\cdot\bar{p}_j^S + \tfrac{r}{n}\cdot\sum_{k\in N}\bar{p}_k = \tfrac{1}{2}\cdot\left[V - (1+r)\cdot c_j + \tfrac{r\cdot s}{n}\cdot\bar{c}_S\right]$$

Consequently, by (3.2.5), the coalitional profit reduces as follows:

$$
\begin{aligned}
& v_\beta(S) = \pi_S((\bar{p}_j^S)_{j\in S}, \bar{p}_{N\setminus S}) \\
= \quad & \sum_{j\in S}[\bar{p}_j^S - c_j]\cdot\left[V-(1+r)\cdot\bar{p}_j^S + \frac{r}{n}\cdot\sum_{k\in N}\bar{p}_k\right] \\
= \quad & \frac{1}{4}\cdot\sum_{j\in S}[\bar{p}_S - c_j]\cdot\left[V-(1+r)\cdot c_j + \frac{r\cdot s}{n}\cdot\bar{c}_S\right] \\
= \quad & \frac{s\cdot V^2}{4\cdot h(s)} - \frac{V}{2}\cdot\sum_{j\in S}c_j + \frac{1+r}{4}\cdot\sum_{j\in S}(c_j)^2 - \frac{r}{4\cdot n}\cdot\left[\sum_{j\in S}c_j\right]^2 \qquad (3.2.14)
\end{aligned}
$$

The latter equality is explained as follows by counting for the six combinations.

$$\sum_{j\in S}\bar{p}_S\cdot V = s\cdot V\cdot\bar{p}_S = \frac{s\cdot V^2}{h(s)}$$

$$\sum_{j\in S}\bar{p}_S\cdot(-)(1+r)\cdot c_j = -(1+r)\cdot s\cdot\bar{p}_S\cdot\bar{c}_S$$

$$\sum_{j\in S}\bar{p}_S\cdot\frac{r\cdot s}{n}\cdot\bar{c}_S = \frac{r\cdot s^2}{n}\cdot\bar{c}_S\cdot\bar{p}_S$$

$$\sum_{j\in S}(-c_j)\cdot V = -V\cdot s\cdot\bar{c}_S = -s\cdot h(s)\cdot\bar{c}_S\cdot\bar{p}_S$$

$$\sum_{j\in S}(-c_j)\cdot(-1)\cdot(1+r)\cdot(c_j) = (1+r)\cdot\sum_{j\in S}(c_j)^2$$

$$\sum_{j\in S}(-c_j)\cdot\frac{r\cdot s}{n}\cdot\bar{c}_S = -\frac{r\cdot s^2}{n}\cdot(\bar{c}_S)^2 = -\frac{r}{n}\cdot\left[\sum_{j\in S}c_j\right]^2$$

Concerning the expression $\bar{c}_S\cdot\bar{p}_S$ counting the three contributions yield

$$\left[-(1+r)\cdot s + \frac{r\cdot s^2}{n} - s\cdot h(s)\right]\cdot\bar{c}_S\cdot\bar{p}_S = -2\cdot s\cdot h(s)\cdot\bar{c}_S\cdot\bar{p}_S = -2\cdot s\cdot V\cdot\bar{c}_S$$

as was to be shown. This completes the overall proof. □

Remark 3.1. For the game of the form (3.3.2), (i)We use (3.3.1) instead the classical definition of convexity by which we are blocked in the process to prove the game is convex.
(ii)The role of function $h : h(x) = 1 + r - \frac{r \cdot x}{n}, x \in R$ is important not only because it shortens the expression of the value function, but also in the condition of (3.2.7) and the game expression w_2.

3.3 Convexity of the non-degenerated Bertrand Oligopoly TU-Game

Let $\mathcal{P}(N) = \{S \mid S \subseteq N\}$ denote the *power set* of any finite set N, consisting of all subsets of N. *A cooperative TU game* (N, w) is a pair of the finite player set N and the so-called characteristic function $w : \mathcal{P}(N) \to R$ satisfying $w(\emptyset) = 0$. For instance, consider the n-person game (N, v) of the form (3.2.8) derived from three almost equal marginal costs $c_1 = c - 1$, $c_2 = 1$, $c_3 = c + 1$, and arbitrary marginal costs c_i for other firms $i \in \{4, 5, \ldots, n\}$, $n \geq 4$. This type of oligopoly game is zero-normalized in that $v(\{i\}) = 0$ for all $i \in N$, while $v(\{1, 2\}) = v(\{2, 3\}) = \frac{1}{2}$ and $v(\{1, 3\}) = v(\{1, 2, 3\}) = 2$. Involving player 2's marginal contribution rewards for joining any coalition, player 2 gains nothing for joining the couple $S = \{1, 3\}$ since $v(\{1, 2, 3\}) - v(\{1, 3\}) = 0$, whereas forming a couple with player 1 yields a positive profit amounting $v(\{1, 2\}) - v(\{1\}) = \frac{1}{2}$. In this setting we say the *convexity constraint* $v(\{1, 2\}) - v(\{1\}) \leq v(\{1, 2, 3\}) - v(\{1.3\})$ is violated.
For the development of the solution theory in the field of Cooperative game theory, it is of significant importance to gather knowledge whether a certain subclass of cooperative games satisfies any particular property like convexity. If the convexity applies, one may benefit from various theoretical results about solution concepts, such as shrinking of the multi-valued solution called Pre-kernel to the single-valued concept called Nucleolus, providing the non-emptiness of the multi-valued solution called Core, its regular structure described through increasing marginal contributions of individuals for joining coalitions, and as such, to study the center of the Core by the solution called Shapley value, as studied in the next section. In the present section we aim to prove the convexity property for Bertrand oligopoly games of the form (3.2.7), in spite of the failure for Bertrand oligopoly games of the degenerated

form (3.2.8). The proof technique is based on the decomposition of any non-degenerated Bertrand oligopoly game into the difference of two appropriate games satisfying the convexity property. Generally speaking, the difference game does not inherit the convexity property, but Bertrand oligopoly games do thanks to the existence of special conditions (see Theorem 3.3).
Recall the definition of convex game again. A cooperative game (N, w) is said to satisfy the *convexity (or supermodularity) property* if its characteristic function $w : \mathcal{P}(N) \to R$ satisfies one of the following equivalent conditions (Shapley, 1971):

$$w(S) + w(T) \leq w(S \cup T) + w(S \cap T) \quad \text{for all } S, T \subseteq N.$$

$$w(S \cup \{i\}) - w(S) \leq w(T \cup \{i\}) - w(T)$$

for all $i \in N$ and all $S \subseteq T \subseteq N \backslash \{i\}$.

$$w(S \cup \{i\}) - w(S) \leq w(S \cup \{i, j\}) - w(S \cup \{j\}) \tag{3.3.1}$$

for all $i, j \in N$, $i \neq j$, and all $S \subseteq N \backslash \{i, j\}$.

A cooperative game (N, w) is said to be *non-essential (additive)* if its characteristic function $w : \mathcal{P}(N) \to R$ satisfies $w(S) = \sum_{j \in S} w(\{j\})$ for all $S \subseteq N$, $S \neq \emptyset$. Obviously, non-essential games are convex since all convexity conditions are met as equalities. According to the alternative, equivalent description (3.2.14), any non-degenerated Bertrand oligopoly game of the form (3.2.7) is decomposed into three types of games as follows: for all $S \neq \emptyset$

$$v_\beta(S) = w_1(S) + \tfrac{V^2}{4} \cdot w_2(S) - \tfrac{r}{4 \cdot n} \cdot w_3(S) \tag{3.3.2}$$

$$w_1(S) = \sum_{j \in S} \left[\tfrac{(1+r)}{4} \cdot (c_j)^2 - \tfrac{V}{2} \cdot c_j \right] \tag{3.3.3}$$

$$w_2(S) = \frac{s}{h(s)} = g(S) \tag{3.3.4}$$

$$w_3(S) = \left[\sum_{j \in S} c_j \right]^2 \tag{3.3.5}$$

In fact, the non-essential game (N, w_1) is redundant for the convexity property. The game (N, w_3) is the square of a standard non-essential game. Generally speaking, the square of any non-essential game is convex too because the marginal contribution of a fixed player i with respect to variable coalitions $S \subseteq N\backslash\{i\}$ are non-decreasing with respect to set inclusion, that is, for all $i, j \in N$, $i \neq j$, and all $S \subseteq N\backslash\{i, j\}$, the following holds:

$$\begin{aligned} & w_3(S \cup \{i\}) - w_3(S) \\ = & \left[\sum_{k \in S \cup \{i\}} c_k\right]^2 - \left[\sum_{k \in S} c_k\right]^2 = \left[c_i + \sum_{k \in S} c_k\right]^2 - \left[\sum_{k \in S} c_k\right]^2 \\ = & (c_i)^2 + 2 \cdot c_i \cdot \sum_{k \in S} c_k \quad \text{and similarly,} \end{aligned} \tag{3.3.6}$$

$$\left[w_3(S \cup \{i, j\}) - w_3(S \cup \{j\})\right] = (c_i)^2 + 2c_i \cdot \sum_{k \in S \cup \{j\}} c_k \quad \text{So, we conclude,}$$

$$\left[w_3(S \cup \{i, j\}) - w_3(S \cup \{j\})\right] - \left[w_3(S \cup \{i\}) - w_3(S)\right] = 2 \cdot c_i \cdot c_j \geq 0 \tag{3.3.7}$$

Lemma 3.1. *Given the substitutability parameter $r > 0$, define the real-valued function $g : [0, \frac{1}{r_n}) \to R$ by*

$$g(x) = \frac{1}{(1+r)} \cdot \frac{x}{(1 - r_n \cdot x)} \quad \text{for all } x \in [0, \tfrac{1}{r_n}), \text{ where } \quad r_n = \tfrac{r}{n \cdot (1+r)} \tag{3.3.8}$$

Then the following holds:
(i) $w_2(S) = g(s)$ for all $S \subseteq N$ with size s, $s = 0, 1, 2, \ldots, n$ and $g(n) = n$
(ii) The function $g : R \to R$ is strictly increasing and strictly convex *on the interval $[0, \frac{1}{r_n})$ and consequently, the game (N, w_2) is* strictly convex *in that*

$$g(s+2) - g(s+1) > g(s+1) - g(s) \qquad \text{for all } s = 0, 1, 2, \ldots, n-2 \tag{3.3.9}$$

(iii) The marginal returns of the function g satisfy

$$g(s+1) - g(s) = \frac{1+r}{h(s) \cdot h(s+1)} \tag{3.3.10}$$

(iv) Applying (3.3.10) twice yields

$$\left[g(s+2) - g(s+1)\right] - \left[g(s+1) - g(s)\right] = \frac{2}{n} \cdot \frac{r \cdot (1+r)}{h(s) \cdot h(s+1) \cdot h(s+2)} \tag{3.3.11}$$

(v) An alternative representation of the worth $v_\beta(S)$ of coalition S is as follows:

$$v_\beta(S) = \frac{1}{4\cdot g(s)}\cdot\Big[V\cdot g(s)-s\cdot\bar{c}_S\Big]^2+\frac{1+r}{4}\cdot\sum_{j\in S}\cdot\Big[c_j-\bar{c}_S\Big]^2 \qquad \textit{provided} \quad \frac{g(s)}{s}>\frac{\bar{c}_S}{V} \tag{3.3.12}$$

Proof. Let $S \subseteq N$ be of size s, $s = 0, 1, 2, \ldots, n$. From (3.3.8) and (3.3.4), we derive

$$g(s) = \frac{s}{(1+r)\cdot(1-r_n\cdot s)} = \frac{s}{(1+r)\cdot(1-\frac{r\cdot s}{n\cdot(1+r)})} = \frac{s}{h(s)} = w_2(S)$$

The representation (3.3.12) of the worth $v_\beta(S)$ agrees with (3.2.14) since

$$\frac{[s\cdot\bar{c}_S]^2}{g(s)} - (1+r)\cdot s\cdot(\bar{c}_S)^2 = \left[\frac{h(s)}{s}-\frac{(1+r)}{s}\right]\cdot\Big[s\cdot\bar{c}_S\Big]^2 = \frac{-r}{n}\cdot\left[\sum_{j\in S}c_j\right]^2$$

It is very simple to verify that the first and second derivative of the differentiable function $g(x)$ are given by $g'(x) = \frac{1}{(1+r)}\cdot\frac{1}{(1-r_n\cdot x)^2} > 0$ as well as $g''(x) = \frac{2}{(1+r)}\cdot\frac{r_n}{(1-r_n\cdot x)^3} > 0$. Recall that a differentiable function is convex if and only if the second derivative is non-negative. □

In summary, so far, all three games (N, w_k), $k = 1, 2, 3$, are convex where the first non-essential game is redundant for the convexity property. Because the non-degenerated Bertrand Oligopoly Game of the form (3.3.2) is the difference of two convex games, it may fail to be convex itself. According to the proof of the next main theorem, convexity still holds for the Bertrand oligopoly game due to the existence of the underlying constraints.

Theorem 3.3. *Suppose the demand's intercept V is large enough to cover any slightly adapted marginal cost in that $V > h(1)\cdot\max_{k\in N} c_k$. Then the non-degenerated Bertrand oligopoly game (N, v_β) of the form (3.3.2) is convex (supermodular).*

Proof. In view of the decomposition (3.3.2), the non-degenerated Bertrand oligopoly game is convex if and only if the game $(N, w_2 - \frac{r}{n\cdot V^2}\cdot w_3)$ is convex. By Lemma 3.1(i), $w_2(S) = g(s)$ for all $S \subseteq N$, whereas (3.3.7) holds in the setting of the game (N, w_3). In summary, the convexity property (3.3.1) applies to the Bertrand oligopoly game if and only if the following holds: for all $i, j \in N$, $i \neq j$, and all $s = 0, 1, 2, \ldots, n-2$,

$$\Big[g(s+2) - g(s+1)\Big] - \Big[g(s+1) - g(s)\Big] \geq \frac{r}{n \cdot V^2} \cdot \Big[2 \cdot c_i \cdot c_j\Big] \quad (3.3.13)$$

By assumption, $c_k \leq \frac{V}{h(1)}$ for $k \in \{i, j\}$ and thus, it suffices to prove

$$\Big[g(s+2) - g(s+1)\Big] - \Big[g(s+1) - g(s)\Big] \geq \frac{2 \cdot r}{n} \cdot \frac{1}{(h(1))^2} \quad (3.3.14)$$

Recall the function $g(x) = \frac{x}{h(x)}$ and the results (3.3.10)–(3.3.11). Note that the expression at the right hand of (3.3.11) is non-decreasing in the variable coalition size s, attaining its minimum at $s = 0$. It follows that

$$\begin{aligned} & \Big[g(s+2) - g(s+1)\Big] - \Big[g(s+1) - g(s)\Big] \\ = \; & \frac{2}{n} \cdot \frac{r \cdot (1+r)}{h(s) \cdot h(s+1) \cdot h(s+2)} \geq \frac{2}{n} \cdot \frac{r \cdot (1+r)}{h(0) \cdot h(1) \cdot h(2)} \\ = \; & \frac{2}{n} \cdot \frac{r}{h(1) \cdot h(2)} \geq \frac{2}{n} \cdot \frac{r}{(h(1))^2} \end{aligned}$$

This completes the proof of convexity for the non-degenerated Bertrand oligopoly game assuming $V > h(1) \cdot \max_{k \in N} c_k$. □

3.4 Shapley value of the non-degenerated Bertrand Oligopoly Game

The decomposition (3.3.2) of the non-degenerated Bertrand oligopoly game (N, v_β) into three types of games permits to determine its *Shapley value* $Sh(N, v_\beta)$ on the basis of the following four properties: linearity, efficiency, symmetry, and strategic equivalence. Generally speaking, the Shapley value $Sh(N, w) = (Sh_i(N, w))_{i \in N}$ of a cooperative game (N, w) is given by an appropriate weighted, probabilistic sum of player's marginal contributions of the form $w(S \cup \{i\}) - w(S)$, $S \subseteq N \backslash \{i\}$, that is [56]

$$Sh_i(N, w) = \sum_{S \subseteq N \backslash \{i\}} p_n(s) \cdot \Big[w(S \cup \{i\}) - w(S)\Big] \qquad \text{for all } i \in N, \quad (3.4.1)$$

where $p_n(s) = \frac{1}{n \cdot \binom{n-1}{s}}$ for all $s = 0, 1, 2, \dots, n-1$.

Due to its probabilistic interpretation, the Shapley value of any non-essential game (N, w) equals the individual worth $w(\{i\})$, $i \in N$. Secondly, because of anonymity and efficiency, the Shapley value of the symmetric game (N, w_2) is fully determined by

$$Sh_i(N, \tfrac{V^2}{4} \cdot w_2) = \tfrac{V^2}{4} \cdot \tfrac{w_2(N)}{n} = \tfrac{V^2}{4 \cdot n} \cdot \tfrac{n}{h(n)} = \tfrac{V^2}{4 \cdot h(n)} = \tfrac{V^2}{4} \quad \text{for all } i \in N.$$

Thirdly, the computation of the Shapley value of the quadratic game (N, w_3) proceeds by using (3.3.5) yielding for all $i \in N$

$$Sh_i(N, w_3) = \sum_{S \subseteq N \setminus \{i\}} p_n(s) \cdot \left[(c_i)^2 + 2 \cdot c_i \cdot \sum_{j \in S} c_j \right] = (c_i)^2 + 2 \cdot c_i \cdot \sum_{S \subseteq N \setminus \{i\}} p_n(s) \sum_{j \in S} c_j$$

Through the inverse order $\sum_{j \in N \setminus \{i\}} c_j \cdot \sum_{\substack{S \subseteq N \setminus \{i\}, \\ S \ni j}} p_n(s)$, together with two combinatorial steps, we arrive at $Sh_i(N, w_3) = c_i \cdot \sum_{j \in N} c_j = n \cdot c_i \cdot \bar{c}_N$. Finally, by applying the linearity of the Shapley value, we conclude the following:

Theorem 3.4. *The Shapley value of the non-degenerated Bertrand oligopoly game of the form (3.3.2) is given by*

$$Sh_i(N, v_\beta) = \tfrac{1+r}{4} \cdot (c_i)^2 - \tfrac{V}{2} \cdot c_i + \tfrac{V^2}{4} - \tfrac{r}{4} \cdot c_i \cdot \bar{c}_N \qquad \textit{for all } i \in N \quad (3.4.2)$$

In short, $\quad Sh_i(N, v_\beta) = \frac{(V - c_i)^2}{4} + \frac{r}{4} \cdot c_i \cdot (c_i - \bar{c}_N) \qquad$ *for all* $i \in N$.

In words, the Shapley value of the non-degenerated Bertrand oligopoly game involves two types of payoffs to each firm i, $i \in N$, namely the square of the netto demand intercept $V - c_i$, as well as a proportional part c_i of the firm's deviation $c_i - \bar{c}_N$ from the average grand coalitional cost.

Without going into details, the multi-valued solution concept called Core is a convex, compact (possibly empty) set of R^n and as such, the Core is the convex hull of its extreme points. According to the theory for strict convex games, developed by Shapley [54], there exist $n!$ distinct extreme points $\vec{x} \in R^n$, of which each component x_i, $i \in N$, equals some marginal contribution of the specific form $x_i = w(P_i^\theta \cup \{i\}) - w(P_i^\theta)$ where $P_i^\theta = \{j \in N \mid \theta(j) < \theta(i)\}$ represents the predecessors of player i with respect to the permutation $\theta : N \to N$ of the player set N. In the framework of Bertrand oligopoly games

(N, v_β), from its decomposition (3.3.2) we conclude that the payoff to player i according to an extreme point of the Core is given as follows (here $S = P_i^\theta$ and s its cardinality):

$$\begin{aligned} x_i &= \tfrac{(1+r)}{4} \cdot (c_i)^2 - \tfrac{V}{2} \cdot c_i + \tfrac{V^2}{4} \cdot \Big[g(s+1) - g(s) \Big] - \tfrac{r}{4 \cdot n} \cdot \Big[(c_i)^2 + 2 \cdot c_i \cdot \sum_{k \in P_i^\theta} c_k \Big] \\ &= \tfrac{h(1)}{4} \cdot (c_i)^2 + \tfrac{V^2}{4} \cdot \tfrac{1+r}{h(s) \cdot h(s+1)} - \tfrac{V}{2} \cdot c_i - \tfrac{r \cdot s}{2 \cdot n} \cdot c_i \cdot \bar{c}_S \end{aligned}$$

For example, if there are no predecessors, that is $S = \emptyset$ and $s = 0$, then the payoff x_i reduces as follows:

$$x_i = \frac{h(1)}{4} \cdot (c_i)^2 + \frac{V^2}{4 \cdot h(1)} - \frac{V}{2} \cdot c_i = \frac{1}{4 \cdot h(1)} \cdot \Big[h(1) \cdot c_i - V \Big]^2 = v_\beta(\{i\})$$

3.5 Concluding Remarks

The Bertrand oligopoly situation with Shubik's demand functions has been modeled as a cooperative TU-game by Lardon [36], but only with reference to identical marginal costs for all firms. The current chapter continues to study the general situation with distinct marginal costs. The complexity of the description of the associated cooperative game, as a result of solving two subsequent optimization programs, is compensated by decomposing the Bertrand oligopoly game into three types of games, namely one non-essential game, one symmetric game, and the square of one non-essential game. Although it concerns the difference of two convex games, it is shown that the Bertrand oligopoly game is convex too. Its current proof technique by decomposition differs from Lardon's proof of convexity for the symmetrical Bertrand oligopoly game which can be found in [35]. The study ends with the computation of the Shapley value and the extreme points of the Core. The $\gamma-$characteristic function form for Bertrand oligopoly games has already been studied in a yet unpublished memorandum [14].

Chapter 4

The Shapley value and the Nucleolus of Service cost savings games

ABSTRACT - In this chapter, the main goal is to introduce the so-called Service cost savings games involving n different customers requiring service provided by companies. For these specific cooperative games, on one hand, we determine the Shapley value allocation for these Service cost savings games through a decomposition method for games into one additive game and one Sharing car pooling cost game, exploiting the linearity of the Shapley value. On the other hand, we determine the Nucleolus allocation as well, by exploiting fully the so-called 1-convexity property for these Service cost savings games.

4.1 Introduction: the Service cost savings game

Consider the situation in which each customer requires some type of transferable service (like information about or transportation of goods). All the local companies provide the same service, but the cost of serving by any company depends on the distance between the locations of the customer $i \in N$ and the company $j \in M$. In the noncooperative setting, for single customer $i \in N$, his total cost will be $\sum_{i \in S} \sum_{j \in M} c_{ij}$, in order to provide service to all the companies M. All the costs of service are gathered in an nonnegative $(n \times m)-$cost matrix $C = [c_{ij}]_{i \in N, j \in M}$, the rows of which are indexed by customers and the columns by companies. In case the service is transferable, any coalition of customers may benefit from their mutual cooperation by providing the service by one company with the cheapest service. In this framework, the Service cost savings game (N, v_C) is defined by its characteristic function $v_C : 2^N \to R$ satisfying $v_C(\emptyset) = 0$ and for all $S \subseteq N, S \neq \emptyset$,

$$v_C(S) = \sum_{i \in S} \sum_{j \in M} c_{ij} - \sum_{j \in M} \min_{i \in S} c_{ij} \tag{4.1.1}$$

Note that the aggregate individual cost of customer i amounts the sum $\sum_{j \in M} c_{ij}$, whereas in the cooperative setting we assume the possibility of transferable services by the members of companies. Hence, the cost of any coalition of customers amounts the sum of aggregate individual cost.

Clearly, one-person coalitions gain nothing and further, the larger the coalition, the larger its gains. So, the Service cost savings game of (4.1.1) is zero-normalized(i.e., $v_C(\{i\}) = 0$ for all $i \in N$) as well as the game is monotonic too(i.e., $v_C(S) \leq v_C(T)$ whenever $S \subseteq T \subseteq N$). By rewriting the order of summation in (4.1.1), we achieve the following decomposition: for all $S \subseteq N, S \neq \emptyset$,

$$v_C(S) = \sum_{j \in M} [\sum_{i \in S} c_{ij} - \min_{i \in S} c_{ij}] = \sum_{j \in M} v_j(S) \tag{4.1.2}$$

By (4.1.2), the game decomposition $v_C = \sum_{j \in M} v_j$ holds. Here, for each $j \in M$, the characteristic function $v_j : 2^N \to R$ of the corresponding game (N, v_j) is given by

$$v_j(S) = \sum_{i \in S} c_{ij} - \min_{i \in S} c_{ij} \quad \text{for all } S \subseteq N, S \neq \emptyset. \tag{4.1.3}$$

Since the first expression at the right hand side of (4.1.3) concerns an additive game, we have that any game $(N, v_j), j \in M$, is relative invariant under strategic equivalence with the cost games (N, C) that is called Sharing car pooling cost game of which the characteristic cost function $C : 2^N \to R$ satisfies

$$C(S) = \min_{k \in S} C(\{k\}) \quad \text{for all } S \subseteq N, S \neq \emptyset. \tag{4.1.4}$$

We remark that any Sharing car pooling cost game (N, C) of the form (4.1.4), in which the minimum operator is replaced by the maximum operator, is known as the airport cost game introduced by Littlechild and Owen [40].

4.2 The Shapley value of the Sharing car pooling cost game and the Service cost savings games through a game decomposition procedure

With the model entitled the Sharing car pooling cost game at hand, we aim to determine the solution part (Shapley value, Nucleolus) through the determination of the well-known Shapley value [56] for Sharing car pooling cost game. Usually, the cost allocation according to the Shapley value provides a distribution of the overall cost amounting $C(N)$ among all the participants based on marginal contributions of the form $C(S \cup \{i\}) - C(S), S \subseteq N \backslash \{i\}, i \in N$. In the framework of the Sharing car pooling cost game, it is rather complex to determine these marginal contributions. Therefore, we proceed by using another algebraic technique based on the (simple) computation of the Shapley value for a certain basis of the game space with fixed player set N. For that purpose, with every coalition $T \subseteq N, T \neq N, T \neq \emptyset$, there is associated the complementary unanimity cost game (N, U_T) given by

$$U_T(S) = \begin{cases} 1, & \text{if } S \neq \emptyset \text{ and } S \cap T = \emptyset; \\ 0, & \text{if } S = \emptyset \text{ or } S \cap T \neq \emptyset. \end{cases} \tag{4.2.1}$$

Further, the complimentary unanimity cost game $(N, U_\emptyset)$ is given by $U_\emptyset(\emptyset) = 0$ and $U_\emptyset(S) = 1$ otherwise. Note that $U_T(N) = 0$ for all $T \subsetneq N$, except $T = \emptyset$. As is shown in [10], the well known Shapley value cost allocation charged to the players of any n-person complementary unanimity cost game (N, U_T) agrees with the semi-egalitarian rule such that members of T receive less the unitary amount, that is,

$$Sh_i(N,U_T) = \begin{cases} \frac{1}{n}, & \text{for all } i \in N\backslash T; \\ \frac{1}{n} - \frac{1}{|T|}, & \text{for all } i \in T. \end{cases} \tag{4.2.2}$$

Theorem 4.1. *Suppose without loss of generality* $0 \leq C(\{1\}) \leq \ldots \leq C(\{n\})$. *Write* $C(\{0\}) = 0$
(i) Every Sharing car pooling cost game (N, C) *of the form (4.1.4) can be decomposed as the following linear combination of a number of complementary unanimity cost games with nonnegative coefficients:*

$$C = \sum_{j=0}^{n-1} \Big[C(\{j+1\}) - C(\{j\}) \Big] \cdot U_{I_j} \tag{4.2.3}$$

where $I_0 = \emptyset$, $I_j = \{1, 2, \ldots, j\}$ *for all* $j \in N$.
(ii) The Shapley cost allocation $Sh(N, C)$ *for the Sharing car pooling cost game* (N, C) *equals*

$$\begin{aligned} Sh_i(N,C) &= \frac{C(\{n\})}{n} - \sum_{j=i}^{n-1} \frac{C(\{j+1\}) - C(\{j\})}{j} \\ &= \frac{C(\{i\})}{i} - \sum_{k=i+1}^{n} \frac{C(\{k\})}{k \cdot (k-1)} \quad \textit{for all} \quad i \in N. \end{aligned} \tag{4.2.4}$$

(iii) The Shapley cost allocation $Sh(N, C)$ *for an* $n-$*person airport cost game* (N, C) *equals*

$$Sh_i(N,C) = \sum_{j=0}^{i-1} \frac{C(\{j+1\}) - C(\{j\})}{n-j} \quad \textit{for all } i \in N. \tag{4.2.5}$$

Proof. (i) Fix coalition $S \subseteq N$, $S \neq \emptyset$. Write $C(S) = C(\{k\})$ such that $k \in S$ and $\ell \notin S$ for all $1 \leq \ell < k$. Given any $0 \leq j \leq n-1$, the following equivalences hold: $U_{I_j}(S) = 1$ iff $S \cap I_j = \emptyset$ iff $0 \leq j < k$. From this, we derive the validity of (4.2.3).
(ii) The validity of (4.2.4) is thanks to the linearity property of the Shapley value applied to the former decomposition result (4.2.3) and using the Shapley value cost allocations of (4.2.2) as follows. Write $C(\{0\}) = 0$. For all $i \in N$,

it holds

$$\begin{aligned} Sh_i(N,C) &= \sum_{j=0}^{n-1}[C(\{j+1\})-C(\{j\})]\cdot Sh_i(N,U_{I_j}) \\ &= \frac{1}{n}\cdot\sum_{j=0}^{n-1}[C(\{j+1\})-C(\{j\})]-\sum_{j=i}^{n-1}\frac{1}{|I_j|}\cdot[C(\{j+1\})-C(\{j\})] \\ &= \frac{C(\{n\})}{n}-\sum_{j=i}^{n-1}\frac{1}{j}\cdot[C(\{j+1\})-C(\{j\})] \end{aligned}$$

(iii) Because of the relationship $\max_{i\in S} C(\{i\}) = C(\{n\}) - \min_{i\in S}[C(\{n\}) - C(\{i\})]$ for all $S \subseteq N$, $S \neq \emptyset$, every n-person airport cost game with stand-alone costs $C(\{i\})$, $i \in N$, (ordered as an increasing sequence), is associated with a Sharing car pooling cost game with adapted stand-alone costs $C(\{n\})-C(\{i\})$, $i \in N$, to be ordered as an increasing sequence. In this setting, (4.2.5) is a direct consequence of (4.2.3) applied to this latter cost game.
According to the Shapley value of Sharing car pooling cost game, we can understand it as follows: for the ordered car owners with $C(\{1\}) \leq C(\{2\}) \leq \ldots \leq C(\{n\})$, in the beginning, there is only one car owner 1, the cost of the journey for him is $C(\{1\})$; then player 2 is involved which makes the cost of player 1 less and the decreasing amount is $\frac{C(\{2\})}{2}$ while the cost of player 2 is $\frac{C(\{2\})}{2}$; after that, player 3 joins in which makes the cost of players 1 and 2 less and the the total decreasing amount is $\frac{C(\{3\})}{3}$ which is divided equally between players 1 and 2 while the cost of player 3 is $\frac{C(\{3\})}{3}$;...; finally, player n joins in, the cost of him equals $\frac{C(\{n\})}{n}$, while this amount is divided equally among the other $n-1$ players. □

Corollary 4.1. *For fixed $i, j, i \in N, j \in M$, we use $N_{ij} = \{k \in N | c_{kj} < c_{ij} \quad or \quad (c_{kj} = c_{ij} \quad and \quad k \leq i)\}$ to denote the set of companies whose cost is smaller or equal to c_{ij}. Then by Theorem (4.1), we obtain the Shapley value of the Service cost game as follows:*

$$Sh_i(N,v_C) = \sum_{j\in M} c_{ij} - \sum_{j\in M}[\frac{c_{ij}}{|N_{ij}|} - \sum_{k\in N\setminus N_{ij}} \frac{c_{kj}}{|N_{kj}|\cdot(|N_{kj}|-1)}], \quad i \in N \quad (4.2.6)$$

Particularly, it requires $\frac{c_{kj}}{|N_{kj}|\cdot(|N_{kj}|-1)} = 0, k \in N$ if $|N_{kj}| = 1$.

4.3 The Nucleolus of the Sharing car pooling cost game and the Service cost savings game

In addition to the Shapley cost allocation, we aim to determine the Nucleolus cost allocation for Sharing car pooling cost game and the Service cost savings game. We avoid the technical definition of the Nucleolus of an arbitrary cost game (N, C) because a significant property of its characteristic function $C : 2^N \to R$ enables us to complete the Nucleolus(as well as the Core) in a straightforward way.

Definition 4.1. [22], [21], [23] A *cooperative cost game* (N, C) with player set N is said to satisfy the 1-*concavity* property if its characteristic cost function $C : 2^N \to R$ satisfies

$$C(N) \leq C(S) + \sum_{i \in N \setminus S} \Delta_i(N, C) \quad \text{for all } S \subsetneq N,\ S \neq \emptyset, \text{ and} \tag{4.3.1}$$

$$C(N) \geq \sum_{i \in N} \Delta_i(N, C) \quad \text{where } \Delta_i(N, C) = C(N) - C(N \setminus \{i\}) \tag{4.3.2}$$

Condition (4.3.1) requires that the cost $C(N)$ of the formation of the grand coalition N can be covered by any coalitional cost $C(S)$ together with the marginal costs $\Delta_i(N, C), i \in N \setminus S$, of all the complementary players. According to condition (4.3.2), all these marginal costs are weakly insufficient to cover the overall cost $C(N)$.

Theorem 4.2. *(i) In the framework of Sharing car pooling cost game of the form (4.1.4),* $1-$*concavity holds.*
(ii) By the decomposition result, the Service cost savings game is $1-$*convex.*
(iii)The Nucleolus of the Sharing car pooling cost game is:

$$Nu_i = \begin{cases} \frac{C(\{2\})}{n}, & \text{for all} \quad i \in N \setminus \{1\}; \\ \frac{C(\{2\})}{n} - [C(\{2\}) - C(\{1\})], & i = 1. \end{cases} \tag{4.3.3}$$

Proof. (i) Let (N, C) be a Sharing car pooling cost game. Suppose without loss of generality that $0 \leq C(\{1\}) \leq C(\{2\}) \leq \ldots \leq C(\{n\})$. Then it holds $C(N) = C(\{1\}), C(N \setminus \{1\}) = C(\{2\})$, and $C(N \setminus \{k\}) = C(\{1\})$ for all $k \in N \setminus \{1\}$. Concerning the marginal costs, we have $\Delta_k(N, C) = 0$

for all $k \in N\backslash\{1\}$, and $\Delta_1(N,C) = C(\{1\}) - C(\{2\})$. We distinguish two types of coalitions S. In case $1 \in S$, then $\Delta_i(N,C) = 0$ for all $i \in N\backslash S$, whereas $C(S) = C(\{1\}) = C(N)$, thus, the 1-concavity condition (4.3.1) is met as a system of equalities. In case $1 \in N\backslash S$, then (4.3.1) reduces to $C(N) \leq C(S) + C(N) - C(N\backslash\{1\})$ or equivalently, $C(S) \geq C(\{2\})$ and hence, the 1-concavity property holds too if $1 \notin S$. This proof technique illustrates that the largest stand-alone costs $C(\{k\})$, $3 \leq k \leq n$, do not matter for the 1-concavity property as long as their truncation remains above the second smallest stand-alone cost $C(\{2\})$. In this setting, (4.3.2) holds trivially.
(ii)The proof is trivial by (i).
(iii) In the framework of $1-$concave cost games of the form (4.1.4), According to the theory developed for n-person 1-concave cost games (N,C) [21], the so-called *Nucleolus* cost allocation for any Sharing car pooling cost game is given by

$$\begin{aligned} & \Delta_i(N,C) - \frac{1}{n} \cdot [\sum_{k \in N} \Delta_k(N,C) - C(N)] \\ = \quad & \Delta_i(N,C) + \frac{C(\{2\})}{n} = \frac{C(\{2\})}{n} \quad \text{for all} \quad i \neq 1. \end{aligned} \tag{4.3.4}$$

So every player receives the egalitarian split $\frac{C(\{2\})}{n}$, and player 1 looses the additional amount $C(\{2\}) - C(\{1\})$. □

Corollary 4.2. *For fixed $j \in M$, we use C_j^1 and C_j^2 to denote the smallest and second smallest cost. Then the Nucleolus of the Service cost savings game is: $Nu_i(N, v_C) = \sum_{j \in M} A_i(j)$ where*

$$A_i(j) = \begin{cases} \frac{C_j^2}{n} + (C_j^1 - C_j^2), & \textit{if} \quad c_{ij} = C_j^1; \\ \frac{C_j^2}{n}, & \textit{otherwise}. \end{cases} \tag{4.3.5}$$

4.4 Concluding Remarks

The proof of the 1-concavity property for Sharing car pooling cost game is treated in Section 4.3, which is the basis for the proof of the 1-convexity of Service cost savings game. The decomposition of the Service cost savings game is an extremely helpful tool for the determination of the Shapley and

Nucleolus cost allocation, which can be found in Section 4.2 and Section 4.3. Due to 1-concavity, the formula (4.3.4) for the Nucleolus cost allocation is fully determined in terms of the marginal costs $\Delta_i(N, C) = C(N) - C(N \backslash \{i\})$, $i \in N$, together with $C(N) = 0$. Finally, two other applications of one-concavity or one-convexity, called library game and co-insurance game respectively, have been studied by Theo Driessen. The Nucleolus for 2-convex games is treated in [15]. The search for other appealing classes of cost games satisfying the 1-concavity property is still going on.

Chapter 5

The Nucleolus for 2-convex TU games

ABSTRACT - In this chapter, we consider 2*-convex* n *person cooperative TU games. The Nucleolus is determined as some type of constrained equal award rule. Its proof is based on Maschler, Peleg, and Shapley's geometrical characterization for the intersection of the Prekernel with the Core. Pairwise bargaining ranges within the Core are required to be in equilibrium. This system of non-linear equations is solved and its unique solution agrees with the Nucleolus.*

5.1 Introduction

For 2-convex n-person cooperative TU games, the Nucleolus is determined as some type of constrained equal award rule. Its proof is based on Maschler, Peleg, and Shapley's geometrical characterization for the intersection of the Pre-kernel with the Core. Pairwise bargaining ranges within the Core are required to be in equilibrium. This system of non-linear equations is solved and its unique solution agrees with the Nucleolus.

Fix the player set N and its power set $\mathcal{P}(N) = \{S|S \subseteq N\}$ consisting of all the subsets of N (including the empty set $\emptyset$). A *cooperative transferable utility*

(TU) game is given by the so-called *characteristic function* $v : \mathcal{P}(N) \to R$ satisfying $v(\emptyset) = 0$. That is, the TU game v assigns to each coalition $S \subseteq N$ its *worth* $v(S)$ amounting the (monetary) benefits achieved by cooperation among the members of S. The *marginal benefit* b_i^v of player i in the game v is defined by $b_i^v = v(N) - v(N \backslash \{i\})$ for all $i \in N$. Associated with the game v there is the so-called *gap function* $g^v : \mathcal{P}(N) \to R$ such that, for every coalition S, its gap $g^v(S)$ represents the surplus of the marginal benefits of its members over its worth, i.e., $g^v(S) = \sum_{k \in S} b_k^v - v(S)$ for all $S \subseteq N$, where $g^v(\emptyset) = 0$. A payoff vector $\vec{x} = (x_k)_{k \in N} \in R^N$ is said to belong to the *Core* $Core(v)$ if it satisfies, besides the efficiency constraint $\sum_{k \in N} x_k = v(N)$, the group rationality constraints $\sum_{k \in S} x_k \geq v(S)$ for all $S \subseteq N$, $S \neq \emptyset$. It is simple to observe that the marginal benefit of any player is an upper bound for Core allocations in that $x_i \leq b_i^v$ for all $i \in N$, all $\vec{x} \in Core(v)$.
Recall the definition of 1-convex game which has been given in chapter 1.

Definition 5.1. An n-person game v is said to be 1-*convex* if its corresponding nonnegative gap function g^v attains its minimum at the grand coalition N, i.e.,

$$g^v(S) \geq g^v(N) \geq 0 \qquad \text{for all } S \subseteq N,\ S \neq \emptyset \tag{5.1.1}$$

In terms of the characteristic function v, (5.1.1) requires that $v(N) \geq v(S) + \sum_{k \in N \backslash S} b_k^v$ for every non-trivial coalition. In words, concerning the division problem, the worth $v(N)$ is sufficiently large to meet the coalitional demand amounting its worth $v(S)$, as well as the desirable marginal benefit for any nonmember of S. The theory on 1-convex n-person games has been well developed [21]. The key feature of 1-convex n-person games is the geometrically regular structure of its Core, composed as the convex hull of n extreme points of which all the coordinates, except one, agree with the marginal benefits of all, but one, players. Moreover, the center of gravity of the Core turns out to coincide with the so-called *Nucleolus* of the 1-convex game. So, the payoff to player i according to the Nucleolus of 1-convex n-person games equals $b_i^v - \frac{g^v(N)}{n}$ for all $i \in N$. Particularly, the Nucleolus on the class of 1-convex n-person games satisfies the mathematically attractive additivity property.

For any payoff vector $\vec{x} \in R^N$ satisfying $\sum_{k \in N} x_k = v(N)$ as well as $x_i \leq b_i^v$ for all $i \in N$, it is simple to observe the validity of the Core constraint

$\sum_{k\in S} x_k \geq v(S)$ whenever the gap of S weakly majorizes the gap of N, i.e., $g^v(S) \geq g^v(N)$. Consequently, for 1-convex n-person games v, the following Core equivalence holds:

$$\vec{x} \in Core(v) \quad \text{if and only if} \quad \sum_{k\in N} x_k = v(N), \qquad x_i \leq b_i^v \quad \text{for all } i \in N \tag{5.1.2}$$

Definition 5.2. An n-person game v is said to be 2-*convex* if on the one hand, the gap of the grand coalition N is weakly majorized by the gap of every multi-person coalition S, and on the other, the *concavity* of the gap function g^v with respect to the sequential formation of the grand coalition N by individuals up to size 1, whereas the remaining $n-1$ players merge as one syndicate to complete the sequential formation of N, i.e.,

$$g^v(S) \geq g^v(N) \qquad \text{for all } S \subseteq N \text{ with } |S| \geq 2, \text{ and} \tag{5.1.3}$$

$$g^v(\{j\}) \geq g^v(N) - g^v(\{i\}) \geq 0 \qquad \text{for all } i,j \in N,\ i \neq j, \text{ or equivalently,} \tag{5.1.4}$$

$$g^v(\{j\}) + g^v(\{i\}) \geq g^v(N) \geq g^v(\{i\}) \qquad \text{for every pair } i,j \in N \text{ of players.} \tag{5.1.5}$$

In view of (5.1.3), for 2-convex n-person games v, the following Core equivalence holds:

$$\vec{x} \in Core(v) \quad \text{if and only if} \quad \sum_{k\in N} x_k = v(N), \quad v(\{i\}) \leq x_i \leq b_i^v \quad \text{for all } i \in N \tag{5.1.6}$$

Alternatively, for 2-convex n-person games, its Core coincides with a so-called *Core catcher* associated with appropriately chosen lower- and upper Core bounds. Our main goal is to exploit the Core equivalence (5.1.6) in order to determine the Nucleolus based on bargaining ranges within the Core.

Example 5.1. Consider the zero-normalized 3-person game $\langle\{1,2,3\},v\rangle$ of which the characteristic function is given by $v(\{1,2\}) = 6$, $v(\{1,3\}) = 7$,

$v(\{2,3\}) = 8$, and $v(N)$ not yet specified.
In case the worth $v(N)$ is small enough, for instance $v(N) = 12$, then the marginal benefit vector $\vec{b}^v = (4,5,6)$, and so, its gap function g^v is given by $g^v(\{i\}) = 4,5,6$ for $i = 1,2,3$, respectively, whereas $g^v(S) = 3$ otherwise. By (5.1.1), the 3-person game v is 1-convex, but fails to be 2-convex, and its Core is the convex hull of the three vertices $(1,5,6)$, $(4,2,6)$, $(4,5,3)$. Further, the Nucleolus coincides with the center $(3,4,5)$ of gravity of the Core.
In case the worth $v(N)$ is large enough, say $v(N) = 15$, then $\vec{b}^v = (7,8,9)$, and so, $g^v(\{i\}) = 7,8,9$ for $i = 1,2,3$, respectively, whereas $g^v(S) = 9$ otherwise. By (5.1.5), the 3-person game v is 2-convex, but fails to be 1-convex, and its Core is the convex hull of the five vertices $(7,0,8)$, $(6,0,9)$, $(0,6,9)$, $(0,8,7)$, $(7,8,0)$ (the latter with geometric multiplicity 2).
In summary, the 3-person game v turns out to be 1-convex iff $10.5 \leq v(N) \leq 13$ and moreover, to be 2-convex iff $v(N) \geq 15$. Appealing examples of 1-convex games are discovered, like the *library game* together with a suitably chosen basis [18] as well as the *co-insurance game* [17]. It is still an outstanding challenge to search for appealing examples of 2-convex games.

5.2 The Nucleolus of 2-convex n-person games

The main purpose is to apply the geometric characterization for the intersection of the Pre-kernel with the Core as introduced by Maschler, Peleg, and Shapley (1979, [43]).

Theorem 5.1. *The Nucleolus of a 2-convex n-person game v is of the parametric form (5.2.3) or (5.2.4), a so-called* constrained equal award rule*, incorporating the constraints amounting a half of the individual gaps $g^v(\{k\})$, $k \in N$. For instance, by (5.2.3), the payoff to any player i according to the Nucleolus equals either the midpoint of its individual worth $v(\{i\})$ and its marginal benefit b_i^v, or its parametric shortage $b_i^v - \lambda$, whichever is more. By (5.2.4), its payoff equals either the same midpoint, or its parametric gain $v(\{i\}) + \lambda$, whichever is less.*

Proof. In view of the Core equivalence (5.1.6) for 2-convex games, the largest amount that can be transferred from player i to another player j with respect to a given Core allocation $\vec{x} \in Core(v)$ while remaining in the Core of the game is either player's i-th decrease amounting $x_i - v(\{i\})$, or player's j-th increase amounting $b_j^v - x_j$, whichever is less. Hence, the largest transfer from

i to j equals $\delta^v_{ij}(\vec{x}) = \min\Big[x_i - v(\{i\}), \quad b^v_j - x_j\Big]$. We are looking for Core allocations $\vec{x}$ satisfying the equilibrium condition $\delta^v_{ij}(\vec{x}) = \delta^v_{ji}(\vec{x})$ for every pair $i, j \in N$ of players.
Define the vector $\vec{y} = (y_k)_{k\in N} \in R^N$ by $y_k = b^v_k - x_k$ for all $k \in N$. Note that $\sum_{k\in N} y_k = g^v(N)$ and the equilibrium conditions may be rewritten by

$$\min\Big[g^v(\{i\}) - y_i, y_j\Big] = \min\Big[g^v(\{j\}) - y_j, y_i\Big] \quad \text{or equivalently,} \tag{5.2.1}$$

$$y_j + \min\Big[g^v(\{i\}), y_i + y_j\Big] = y_i + \min\Big[g^v(\{j\}), y_i + y_j\Big] \tag{5.2.2}$$

for every pair of players.

From (5.2.2), it follows that $y_j \geq y_i$ whenever $g^v(\{j\}) \geq g^v(\{i\})$. In fact, the system (5.2.1) of pairwise (non-linear) equations, together with the adapted efficiency constraint $\sum_{k\in N} y_k = g^v(N)$, is uniquely solvable [20] (page 47) and its unique solution is of the *parametric* form

$$y_k = \min\Big[\lambda, \frac{g^v(\{k\})}{2}\Big] \quad \text{and so,} \quad x_k = v(\{k\}) + \max\Big[g^v(\{k\}) - \lambda, \frac{g^v(\{k\})}{2}\Big] \tag{5.2.3}$$

for all $k \in N$, where the parameter $\lambda \in R$ is determined by the efficient constraints $\sum_{k\in N} y_k = g^v(N)$ and $\sum_{k\in N} x_k = v(N)$. The latter solution (5.2.3) applies only if $\frac{1}{2} \cdot \sum_{k\in N} g^v(\{k\}) \geq g^v(N)$, otherwise for all $k \in N$

$$y_k = \max\Big[g^v(\{k\}) - \lambda, \frac{g^v(\{k\})}{2}\Big] \quad \text{and so,} \quad x_k = v(\{k\}) + \min\Big[\lambda, \frac{g^v(\{k\})}{2}\Big] \tag{5.2.4}$$

□

Remark 5.1. The non-void intersection of the two classes of 1-convex and 2-convex n-person games is fully characterized by identical individual gaps such that $g^v(\{k\}) = g^v(N)$ for all $k \in N$. In this setting, (5.2.3) applies, and the parameter λ is determined through the slightly adapted efficiency constraint

$$\sum_{k\in N} \min[\lambda, \frac{g^v(N)}{2}] = g^v(N). \qquad \text{Thus,} \qquad y_k = \lambda = \frac{g^v(N)}{n} \quad \text{and so,}$$

the Nucleolus payoff equals $x_k = b_k^v - y_k = b_k^v - \frac{g^v(N)}{n}$ for all $k \in N$, which is in accordance with previous remarks involving the Nucleolus payoff vector $\vec{x}$.

Remark 5.2. In view of the Core equivalence (5.1.2) for 1-convex n-person games v, the largest transfer from player i to another player j, while remaining in the Core of the game, is fully determined by player's j-th increase amounting $b_j^v - x_j$. That is, $\delta_{ij}^v(\vec{x}) = b_j^v - x_j$ for all $i, j \in N$, $i \neq j$. The equilibrium condition $\delta_{ij}^v(\vec{x}) = \delta_{ji}^v(\vec{x})$, or equivalently, the system of linear equations $b_j^v - x_j = b_i^v - x_i$ for every pair $i, j \in N$ of players, is easily solved by the unique efficient payoff vector of which the coordinates are given by $b_k^v - \frac{g^v(N)}{n}$, $k \in N$.

Remark 5.3. In [41], the authors study the so-called class of *compromise stable* games of which the Core agrees with a certain Core cover in the sense of (5.1.6) by replacing the weak lower bound $v(\{i\})$ by another sharp lower bound amounting $b_i^v - \min_{S \ni i} g^v(S)$. Their approach to determine the Nucleolus of compromise stable games is totally different and strongly based on the study of (convex) bankruptcy games [41](Theorem 4.2, pages 497-498). Our geometrical approach to determine the Nucleolus of compromise stable applies once again, but is left to the reader. In fact, (5.2.1) applies once again, replacing $g^v(\{i\})$ by $\min_{S \ni i} g^v(S)$.

Chapter 6

A new characterization of the pre-kernel for TU games through its indirect function and its application to determine the Nucleolus for 1-convex and 2-convex games

ABSTRACT - In this chapter, the main goal is twofold. Thanks to the so-called indirect function known as the dual representation of the characteristic function of a coalitional TU game, we derive a new characterization of the Pre-kernel of the coalitional game using the evaluation of its indirect function on the tails of pairwise bargaining ranges arising from a given payoff vector. Secondly, we study two subclasses of coalitional games of which its indirect function has an explicit formula and show the applicability of the determination of the Pre-kernel (Nucleolus) for such types of games using the indirect function. Two such subclasses of games concern the $1-$*convex and* 2*-convex* n *person games.*

6.1 Introduction and Notions

As shown in [**?**] [17], certain practical problems such as co-insurance situations and library situations can be modeled as a cooperative game in characteristic function form. Formally, a cooperative game on player set N is a characteristic function $v : \mathcal{P}(N) \to R$ defined on $\mathcal{P}(N)$ satisfying $v(\emptyset) = 0$. Here $\mathcal{P}(N)$ denotes the power set of the finite player set N, given by $\mathcal{P}(N) = \{S | S \subseteq N\}$, and shortly called a game v on N. In [42], the dual representation of cooperative games based on Fenchel Moreau Conjugation has been introduced, with every game v on N, there is associated the indirect function $\pi^v : R^N \to R$, given by

$$\pi^v(\vec{y}) = \max_{S \subseteq N} e^v(S, \vec{y}) \qquad \text{for all } \vec{y} = (y_k)_{k \in N} \in R^N, \tag{6.1.1}$$

The excess $e^v(S, \vec{y})$ of a non-empty coalition S at the *salary vector* $\vec{y}$ in the game v represents the net profit the (unique) employer would receive from the selection of coalition S, assuming the members of S will produce, using the resources that are available to the employer, a total amount of output the monetary utility of which is measured by $v(S)$, and the (possibly negative) salary required by the player i amounts y_i, $i \in N$. Write $e^v(\emptyset, \vec{y}) = 0$. In the game theory setting, the efficient salary vectors of which all the excess are non-positive, compose the multi-valued solution concept called Core, that is

$$Core(v) = \{\vec{y} \in R^N | e^v(N, \vec{y}) = 0, e^v(S, \vec{y}) \leq 0 \quad \text{for all } S \subseteq N,\ S \neq \emptyset\}, \tag{6.1.2}$$

According to [42], the indirect function $\pi^v : R^N \to R$ of a game v on N is a non-increasing convex function which attains its minimum at level zero, i.e., $\min_{\vec{y} \in R^N} \pi^v(\vec{y}) = 0$.

In this chapter, we use indirect function to determine the Nucleolus for two subclasses of games concerning $1-$convex and $2-$convex games [21] [16]. The theory on $1-$convex n-person games has been well developed by Theo Driessen. The key feature of this kind of games is the geometrically regular structure of its Core. For $2-$convex games, its Core coincides with a so-called Core catcher associated with appropriately chosen lower and upper Core bounds.

6.2 The indirect function of 1-convex and 2-convex n person games

Given a game (N, v), its corresponding benefits vector $\vec{b^v} = (b_i^v)_{i \in N}$ is defined by $b_i^v = v(N) - v(N \setminus \{i\})$, $i \in N$. Note that the vector $\vec{b^v}$ is an upper bound for Core allocations in that $y_i \leq b_i^v$ for all $i \in N$, all $\vec{y} \in Core(v)$. In terms of the characteristic function v, the 1-convexity property requires that, concerning the division problem, the worth $v(N)$ is sufficiently large to meet the coalitional demand amounting its worth $v(S)$, as well as the desirable marginal benefit by any individual not belonging to coalition S. For notation sake, write $\vec{z}(T)$ instead of $\sum_{k \in T} z_k$ for any coalition $T \subseteq N$ and any vector $\vec{z} = (z_k)_{k \in N} \in R^N$, where $\vec{z}(\emptyset) = 0$, and use $\vec{y} \leq \vec{b^v}$ instead of $y_i \leq b_i^v$ for all $i \in N$.

Definition 6.1. A game v on N is said to be 1-*convex* if it holds

$$\sum_{k \in N} b_k^v \geq v(N) \quad \text{and} \quad v(N) \geq v(S) + \sum_{k \in N \setminus S} b_k^v \qquad \text{for all } S \subseteq N,\, S \neq \emptyset. \tag{6.2.1}$$

Example 6.1. Let the three-person game v on $N = \{1, 2, 3\}$ be given by $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 1$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 6$, $v(\{2, 3\}) = 7$, $v(N) = 10$. It is left to the reader to check the 1-convexity of this game using the marginal benefit vector $b^v = (3, 4, 6)$. It turns out that the Core coincides with the triangle with the three vertices $(0, 4, 6)$, $(3, 1, 6)$, $(3, 4, 3)$. In fact, $(y_1, y_2, y_3) \in Core(v)$ is equivalent to $y_1 + y_2 + y_3 = 10$ and $y_1 \leq 3, y_2 \leq 4, y_3 \leq 6$. Under the latter upper Core bound assumption $y \leq b^v$, the first part of the following theorem reports that the level equation $\pi^v(y) = c$ for its indirect function π^v is solved by the hyperplane equation $y_1 + y_2 + \ldots + y_n = v(N) - c$ provided $c > 0$. Here the larger the strictly positive level c, the smaller $v(N) - c$. In case $c = 0$, then its level equation $\pi^v(y) = 0$ is solved by any hyperplane equation $y_1 + y_2 + \ldots + y_n = d$ where the real number d ranges from $b^v(N)$ to $v(N)$. The lowest hyperplane with $d = v(N)$ represents the Core of the $1-$convex game.

Theorem 6.1. *Let v be a* 1*-convex game on N and we study the indirect function of this game with respect to the following two types of vectors, given $\vec{y} \in R^n$.*
Type 1: $\vec{y} \leq \vec{b^v}$.

Type 2: There exists a unique $\ell \in N$ with $y_\ell > b^v_\ell$ and $y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$. Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$
\begin{aligned}
(i)\pi^v(\vec{y}) &= \max\left[0, \quad v(N) - \sum_{k\in N} y_k\right] \quad \textit{for vectors of type 1.} \\
(ii)\pi^v(\vec{y}) &= \max\left[0, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\right] \\
&= \max\left[0, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b^v_\ell\right] \quad \textit{for vectors of type 2.}
\end{aligned}
$$

Proof. (i) Let $S \subseteq N$, $S \neq \emptyset$, and $\vec{y} \in R^N$ with $y_i \leq b^v_i$ for all $i \in N$. From (6.2.1), we derive

$$
\begin{aligned}
v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + \vec{y}(N\backslash S) \\
&\leq v(S) - \vec{y}(N) + \vec{b}^v(N\backslash S) \leq v(N) - \vec{y}(N) \qquad (6.2.2)
\end{aligned}
$$

Thus, the restriction of the indirect function π^v to the comprehensive hull of the marginal benefit vector $\vec{b}^v$ attains its maximum either for $S = N$ or $S = \emptyset$. (ii) For every $\vec{y} \in R^N$ such that there exists a unique $\ell \in N$ with $y_\ell > b^v_\ell$ and $y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$, it holds that, on the one hand, $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ for all $S \subseteq N$ with $\ell \in S$ because the above chain (6.2.2) of inequalities still holds due to $\ell \notin N\backslash S$. For all $S \subseteq N$ with $\ell \notin S$, it holds

$$
\begin{aligned}
v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + y_\ell + \vec{y}(N\backslash(S \cup \{\ell\})) \\
&\leq v(S) - \vec{y}(N) + y_\ell + \vec{b}^v(N\backslash(S \cup \{\ell\})) \\
&= v(S) - \vec{y}(N) + y_\ell - b^v_\ell + \vec{b}^v(N\backslash S) \\
&\leq v(N) - \vec{y}(N) + y_\ell - b^v_\ell = v(N\backslash\{\ell\}) - \vec{y}(N\backslash\{\ell\}) (6.2.3)
\end{aligned}
$$

In this setting, the indirect function π^v attains its maximum either for $S = N$, $S = N\backslash\{\ell\}$ or $S = \emptyset$, but $S = N$ cancels. □

Corollary 6.1. *For every* 1*-convex game v on N and the payoff vector $\vec{y} = (y_k)_{k\in N} \in R^N$, it holds:*

$$\vec{y} \in Core(v) \Leftrightarrow \vec{y}(N) = v(N), \pi^v(\vec{y}) = 0 \Leftrightarrow \vec{y}(N) = v(N), \vec{y} \leq \vec{b}^v.$$

The former if and only if implication is trivial, while the latter if and only if implication is shown by the (partial) determination of the indirect function for 1-convex games according to Theorem 6.1.

In the remainder of this section, we switch from 1-convex to 2-convex games. In this framework, it is useful to introduce the so-called *gap function* $g^v : \mathcal{P}(N) \to R$ of a game v on N, given by $g^v(S) = \vec{b}^v(S) - v(S)$ for all $S \subseteq N$, $S \neq \emptyset$, and $g^v(\emptyset) = 0$. In view of (6.2.1), a game v on N is 1-convex if and only if the nonnegative gap function attains its minimum at the grand coalition, i.e., $0 \leq g^v(N) \leq g^v(S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Definition 6.2. [21] A game v on N is said to be 2-*convex* if the following two conditions hold:

$$g^v(\{i\}) + g^v(\{j\}) \geq g^v(N) \geq g^v(\{i\}) \quad \text{for any players } i, j \in N,\ i \neq j \tag{6.2.4}$$

$$v(N) \geq v(S) + \sum_{k \in N \setminus S} b_k^v \qquad \text{for all } S \subseteq N,\ |S| \geq 2 \tag{6.2.5}$$

For 2-convexity, the main condition (6.2.1) is kept except for singletons, of which the gap is leveled below the gap of the grand coalition, whereas the sum of two such gaps majorizes the gap of the grand coalition.

Theorem 6.2. *Let v be a 2-convex game on N and we study the indirect function of this game with respect to the following four types of vectors, given $\vec{y} \in R^n$.*
Type 1: $\vec{y} \leq \vec{b}^v$.
Type 2: There exists a unique $\ell \in N$ with $y_\ell > b_\ell^v \geq v(\{\ell\})$ and $v(\{i\}) \leq y_i \leq b_i^v$ for all $i \in N$, $i \neq \ell$.
Type 3: There exists a unique $j \in N$ with $y_j < v(\{j\}) \leq b_j^v$ and $v(\{i\}) \leq y_i \leq b_i^v$ for all $i \in N$, $i \neq j$.
Type 4: There exist unique $j, \ell \in N$ with $y_\ell > b_\ell^v \geq v(\{\ell\})$, $y_i \leq b_i^v$ for all $i \in N$, $i \neq \ell$, and $y_j < v(\{j\}) \leq b_j^v$, $y_i \geq v(\{i\})$ for all $i \in N$, $i \neq j$. Then its

indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$
\begin{aligned}
(i)\pi^v(\vec{y}) &= \max\left[0, v(N) - \sum_{k\in N} y_k, \quad (v(\{i\}) - y_i)_{i\in N}\right] \textit{ for vectors of type 1.}\\
(ii)\pi^v(\vec{y}) &= \max\left[0, v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\right]\\
&= \max\left[0, v(N) - \sum_{k\in N} y_k + y_\ell - b^v_\ell\right] \textit{ for vectors of type 2.}\\
(iii)\pi^v(\vec{y}) &= \max\left[v(N) - \sum_{k\in N} y_k, \quad v(\{j\}) - y_j\right] \textit{ for vectors of type 3.}\\
(iv)\pi^v(\vec{y}) &= \max\left[v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k, \quad v(\{j\}) - y_j\right]\\
&= \max\left[v(N) - \sum_{k\in N} y_k + y_\ell - b^v_\ell, \quad v(\{j\}) - y_j\right] \textit{ for vectors of type 4.}
\end{aligned}
$$

The proof is similar to the previous proof of Theorem(6.1) and is omitted here.

Corollary 6.2. *Let v be a 2-convex game on N and let $\vec{y} = (y_k)_{k\in N} \in R^n$. Then $\vec{y} \in Core(v)$ iff $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$ iff $\vec{y}(N) = v(N)$ and $v(\{i\}) \le y_i \le b^v_i$ for all $i \in N$.*

The former if and only if statement is general and the latter is shown by the structure of the indirect function.

6.3 Solving the Pre-kernel by means of the indirect function

In this section, we characterize the Pre-kernel of a game on N by the evaluation of the indirect function of the game at pairwise bargaining ranges arising from the payoff vector involved. Formally, for every pair of players $i, j \in N, i \neq j$, the surplus $s^v_{ij}(\vec{y})$ of player i against player j at the (salary) vector $\vec{y}$ in the game v on N is given by the maximal excess among coalitions containing player i, but not containing player j. That is,

Definition 6.3. Let v be a game on N and $\vec{y} = (y_k)_{k\in N} \in R^N$.

(i) For every pair of players $i, j \in N$, $i \neq j$, the *surplus* $s^v_{ij}(\vec{y})$ of player i against player j at the (salary) vector $\vec{y}$ in the game v is given by

$$s^v_{ij}(\vec{y}) = \max\left[e^v(S,\vec{y})| \quad S \subseteq N, \quad i \in S, \quad j \notin S\right] \tag{6.3.1}$$

(ii) The *Pre-kernel* $\mathcal{K}^*(v)$ of the game v consist of *efficient* salary vectors of which all the pairwise surpluses are in equilibrium, that is [43]

$$\mathcal{K}^*(v) = \{\vec{y} \in R^N | e^v(N,\vec{y}) = 0, s^v_{ij}(\vec{y}) = s^v_{ji}(\vec{y}) \quad \text{for all } i, j \in N, \, i \neq j.\} \tag{6.3.2}$$

For the alternative description of the Pre-kernel, with every payoff vector $\vec{x} = (x_k)_{k\in N} \in R^N$, every pair of players $i, j \in N$, $i \neq j$, and every transfer amount $\delta \geq 0$ from player i to player j, there is associated the modified payoff vector $\vec{x}^{ij\delta} = (\vec{x}^{ij\delta}_k)_{k\in N} \in R^N$ defined by $x^{ij\delta}_i = x_i - \delta$, $x^{ij\delta}_j = x_j + \delta$, and $x^{ij\delta}_k = x_k$ for all $k \in N\backslash\{i, j\}$.

Theorem 6.3. *Let v be a game on N and $\vec{x} = (x_k)_{k\in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$.*

(i) For every pair of players $i, j \in N$, $i \neq j$, the indirect function $\pi^v : R^N \to R$ satisfies $\pi^v(\vec{x}^{ij\delta}) = s^v_{ij}(\vec{x}) + \delta$, provided $\delta \geq 0$ is sufficiently large.

(ii) $\vec{x} \in \mathcal{K}^*(v)$ if and only if the evaluation of the pairwise bargaining ranges arising from $\vec{x}$ through the indirect function are in equilibrium, that is, for every pair of players $i, j \in N$, $i \neq j$, the indirect function satisfies $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for δ sufficiently large.

Proof. Fix the pair of players $i, j \in N$, $i \neq j$. Firstly, we claim that coalitions not containing player i or containing player j are redundant for maximizing the excesses at the modified payoff vector $\vec{x}^{ij\delta}$, provided the transfer amount $\delta \geq 0$ is sufficiently large. For that purpose, for all coalitions $S \subseteq N\backslash\{i\}$, $T \subseteq N\backslash\{j\}$, note the following two equivalences:

$$v(S\cup\{i\}) - \sum_{k\in S\cup\{i\}} x^{ij\delta}_k \geq v(S) - \sum_{k\in S} x^{ij\delta}_k \quad \text{iff} \quad \delta \geq v(S) - v(S\cup\{i\}) + x_i \tag{6.3.3}$$

$$v(T\cup\{j\}) - \sum_{k\in T\cup\{j\}} x_k^{ij\delta} \le v(T) - \sum_{k\in T} x_k^{ij\delta} \quad \text{iff} \quad \delta \ge v(T\cup\{j\}) - v(T) - x_j \tag{6.3.4}$$

From (6.1.1) and (6.3.3)–(6.3.4) respectively, we derive that

$$\pi^v(\vec{x}^{ij\delta}) = \max_{S\subseteq N}\left[v(S) - \sum_{k\in S} x_k^{ij\delta}\right] = \max_{\substack{S\subseteq N,\\ i\in S,\quad j\notin S}}\left[v(S) - \sum_{k\in S} x_k^{ij\delta}\right] \tag{6.3.5}$$

where the choice of δ can be improved by

$$\delta \ge \max\left[\max_{S\subseteq N\setminus\{i\}} |v(S\cup\{i\}) - v(S) - x_i|, \quad \max_{T\subseteq N\setminus\{j\}} |v(T\cup\{j\}) - v(T) - x_j|\right]$$

because of $|\alpha| \ge \alpha$ as well as $|\alpha| \ge -\alpha$ for all $\alpha \in R$. Finally, from (6.3.5), $x_i^{ij\delta} = x_i - \delta$, and (6.3.1) respectively, we conclude that, for $\delta \ge 0$ sufficiently large, the following chain of equalities holds:

$$\pi^v(\vec{x}^{ij\delta}) = \max_{\substack{S\subseteq N,\\ i\in S,\quad j\notin S}}\left[v(S) - \sum_{k\in S} x_k^{ij\delta}\right] = \max_{\substack{S\subseteq N,\\ i\in S,\quad j\notin S}}\left[v(S) - \sum_{k\in S} x_k\right] + \delta = s_{ij}^v(\vec{x}) + \delta$$

This proves part (i). Together with (6.3.2), part (ii) follows immediately. □

6.4 Remarks about determination of the Nucleolus

The aim of this section is to illustrate the significant role of the indirect function for two classes of games (1-convex and 2-convex) to determine its Nucleolus through a uniform approach replacing its original computation approach. Under these circumstances, the Nucleolus belongs always to the Pre-kernel, and so it is sufficient to solve the system for its unique solution. Thus we avoid the formal definition of the Nucleolus.

Remark 6.1. Suppose the game v on N is 1-convex. For every payoff vector $\vec{x} = (x_k)_{k\in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$ as well as $\vec{x} \le \vec{b}^v$, and for every pair of players $i, j \in N$, $i \ne j$, the evaluation of the indirect function $\pi^v : R^N \to R$ at the tail of the bargaining range described by the corresponding modified payoff vector $\vec{x}^{ij\delta}$ is in accordance with Theorem 6.1(i)–(ii) dependent on the size of its j-th component $\vec{x}_j^{ij\delta} =$

$x_j + \delta$ in comparison to player j-th marginal benefit b_j^v. From the explicit formula for the indirect function of 1-convex games, we conclude the following:

$$\pi^v(\vec{x}^{ij\delta}) = 0 \quad \text{if} \quad x_j^{ij\delta} \leq b_j^v, \text{ that is } \delta \leq b_j^v - x_j$$

$$\pi^v(\vec{x}^{ij\delta}) = \max\Big[0, \quad x_j^{ij\delta} - b_j^v\Big] = x_j + \delta - b_j^v > 0 \quad \text{otherwise}$$

For sufficiently large δ, the equilibrium condition $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ is met if and only if $x_j + \delta - b_j^v = x_i + \delta - b_i^v$, that is $x_j - b_j^v = x_i - b_i^v$ for all $i \neq j$. Together with the efficiency principle $\vec{x}(N) = v(N)$, the unique solution of this system of linear equations is given by

$$x_i = b_i^v - \frac{\alpha}{n} \qquad \text{for all } i \in N, \text{ where} \qquad \alpha = \vec{b}^v(N) - v(N) \geq 0$$

The latter solution is known as the Nucleolus and turns out to coincide with the gravity of the Core being the convex hull of n extreme points of the form $\vec{b}^v - \alpha \cdot \vec{e}_i$, $i \in N$. Here $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denotes the standard basis of R^n.

We consider once again the 3-person game of the Example 6.1 in order to illustrate Remark 6.1 and Theorem 6.3. Let payoff vector $\vec{x}$ satisfy $\vec{x}(N) = v(N) = 10$ as well as $\vec{x} \leq \vec{b}^v = (3, 4, 6)$. From Remark 6.1, we obtain that $\pi^v(\vec{x}^{ij\delta}) = x_j + \delta - b_j^v$, $\pi^v(\vec{x}^{ji\delta}) = x_i + \delta - b_i^v$ for sufficiently large δ. By Theorem 6.3(ii), it holds that $\vec{x} \in \mathcal{K}^*(v)$ iff $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for δ sufficiently large. Thus, $\vec{x} \in \mathcal{K}^*(v)$ iff $x_j + \delta - b_j^v = x_i + \delta - b_i^v$ and due to efficiency, the Nucleolus is given by $\vec{x} = (2, 3, 5)$.

Remark 6.2. Suppose the game v on N is 2-convex. From the explicit formula for the indirect function of 2-convex n-person games, as presented in Theorem 6.2(iv), we conclude that, for $\delta \geq 0$ sufficiently large, the equilibrium condition $\pi^v(\vec{x}^{j\ell\delta}) = \pi^v(\vec{x}^{\ell j\delta})$ reduces to the following system of equations: for every pair of players $j, \ell \in N$, $j \neq \ell$,

$$\min\Big[b_\ell^v - x_\ell, \quad x_j - v(\{j\})\Big] = \min\Big[b_j^v - x_j, \quad x_\ell - v(\{\ell\})\Big]$$

As shown in [16], the unique solution is of the parametric form $x_i = v(\{i\}) + \min\Big[\mu, \quad \frac{b_i^v - v(\{i\})}{2}\Big]$ for all $i \in N$, where the parameter $\mu \in R$ is determined by the efficiency condition $\vec{x}(N) = v(N)$.

Chapter 7

The Indirect Function of Compromise stable TU Games and Clan TU Games as a tool for the determination of its Nucleolus and Pre-kernel

ABSTRACT - In this chapter, we illustrate that the so-called indirect function of a cooperative game in characteristic function form is applicable to determine the Nucleolus for a subclass of coalitional games called compromise stable TU games. In accordance with the Fenchel-Moreau theory on conjugate functions, the indirect function is known as the dual representation of the characteristic function of the coalitional game. The key feature of compromise stable TU games is the coincidence of its Core with a box prescribed by certain upper and lower Core bounds. For the purpose of the determination of the Nucleolus, we benefit from the interrelationship between the indirect function and the Pre-kernel of coalitional TU games. The class of compromise stable TU games contains the subclasses of clan games, big boss games, 1- *and* 2-*convex* n *person TU games. As an adjunct, this chapter reports the indirect function of clan games for the purpose to determine its Nucleolus.*

7.1 Compromise stable TU Games

Fix the finite player set N and its power set $\mathcal{P}(N) = \{S | S \subseteq N\}$ consisting of all the subsets of N (including the empty set $\emptyset$). A *cooperative transferable utility game*, or TU game for short, is given by the so-called *characteristic function* $v : \mathcal{P}(N) \to R$ satisfying $v(\emptyset) = 0$. That is, the TU game v assigns to each coalition $S \subseteq N$ its *worth* $v(S)$ amounting the monetary benefits achieved by cooperation among the members of S.

In the framework of set-valued solution concepts for TU games, we aim to determine the Pre-kernel for a special subclass of TU games called compromise stable TU games [41] using a new mathematical tool called the indirect function [42]. The economic interpretation of this function which can be found in definition 7.1 is the following. An employer has to select among the players those who will provide the maximum profit to him. In case the non-empty coalition $S \subseteq N$ is selected, then its members will provide, using the resources that are available to the employer, a total amount of output the monetary utility of which is represented by the worth $v(S)$. The expression $e^v(S, \vec{y}) = v(S) - \sum_{k \in S} y_k$, called the *excess* of coalition S at the payoff vector $\vec{y} = (y_k)_{k \in N} \in R^N$ in the TU game v, is thus the net profit the employer would obtain from the coalition S if the (possibly negative) salary required by the player i amounts y_i, $i \in N$. Write $e^v(\emptyset, \vec{y}) = 0$. In accordance with the Fenchel-Moreau theory on conjugate functions, the indirect function provides a dual representation to TU games in the sense that indirect functions provide the same information as characteristic functions because a simple formula permits to recover any characteristic function from its associated indirect function.

Definition 7.1. [42] With every TU game $v : \mathcal{P}(N) \to R$, there is associated the *indirect function* $\pi^v : R^N \to R$, given by

$$\pi^v(\vec{y}) = \max_{S \subseteq N} e^v(S, \vec{y}) = \max_{S \subseteq N} \left[v(S) - \sum_{k \in S} y_k \right] \qquad \text{for all } \vec{y} = (y_k)_{k \in N} \in R^N. \tag{7.1.1}$$

Definition 7.2. The *Core(v)* of the TU game $v : \mathcal{P}(N) \to R$ consists of *efficient* salary vectors of which all the excesses are non-positive, that is

$$Core(v) = \{\vec{y} \in R^N | e^v(N, \vec{y}) = 0 \quad \text{and} \quad e^v(S, \vec{y}) \leq 0 \quad \text{for all } S \subsetneq N,\ S \neq \emptyset\}.$$

(7.1.2)

Equivalently, $\vec{y} \in Core(v)$ if and only if $e^v(N, \vec{y}) = 0$ and $\pi^v(\vec{y}) = 0$.

Concerning the definition of compromise stable TU games, we follow the notation as used in [41].

Definition 7.3. Let $v : \mathcal{P}(N) \to R$ be a TU game.

(i) The *utopia demand* vector $\vec{M}^v = (M_k^v)_{k \in N} \in R^N$ is given by $M_i^v = v(N) - v(N \backslash \{i\})$ for all $i \in N$.

(ii) The *minimum right* vector $\vec{m}^v = (m_k^v)_{k \in N} \in R^N$ is given by

$$m_i^v = \max \left[v(S) - \sum_{k \in S \backslash \{i\}} M_k^v \mid \quad S \subseteq N, \quad i \in S \right] \quad \text{for all } i \in N. \tag{7.1.3}$$

(iii) The *Core cover* $CC(v) \subseteq R^N$ consist of efficient payoff vectors representing compromises between utopia demands as well as minimum rights, that is

$$CC(v) = \{ \vec{y} \in R^N | e^v(N, \vec{y}) = 0 \quad \text{and} \quad m_i^v \leq y_i \leq M_i^v \quad \text{for all } i \in N \}. \tag{7.1.4}$$

(iv) The TU game v is called *compromise stable* if $CC(v) = Core(v)$.

We remark that the inclusion $Core(v) \subseteq CC(v)$ holds in general because the utopia demand vector $\vec{M}^v$ and the minimum right vector $\vec{m}^v$ are well-known to be an upper and lower bound for the Core, respectively. As a first main contribution, we provide an alternative proof of the following characterization of compromise stable TU games. For any non-empty coalition $T \subseteq N$ and any payoff vector $\vec{z} = (z_k)_{k \in N} \in R^N$, write $\vec{z}(T) = \sum_{k \in T} z_k$, where $\vec{z}(\emptyset) = 0$.

Theorem 7.1. *[41] A TU game $v : \mathcal{P}(N) \to R$ is compromise stable if and only if*

$$v(S) \leq \max \left[\sum_{k \in S} m_k^v, \quad v(N) - \sum_{k \in N \backslash S} M_k^v \right] \qquad \textit{for all } S \subseteq N,\ S \neq \emptyset. \tag{7.1.5}$$

Alternative proof. (i) Suppose (7.1.5) holds. We prove the coincidence $CC(v) = Core(v)$. It suffices to prove the inclusion $CC(v) \subseteq Core(v)$. Suppose $\vec{y} = (y_k)_{k\in N} \in CC(v)$. Then $m_i^v \leq y_i \leq M_i^v$ for all $i \in N$. Let $S \subseteq N$, $S \neq \emptyset$. Clearly, $\vec{y}(S) \geq \vec{m}^v(S)$, whereas $\vec{y}(S) = v(N) - \vec{y}(N\backslash S) \geq v(N) - \vec{M}^v(N\backslash S)$. Hence, $\vec{y}(S) \geq \max\left[\vec{m}^v(S), \quad v(N) - \vec{M}^v(N\backslash S)\right]$. Due to (7.1.5), $\vec{y}(S) \geq v(S)$ for all $S \subseteq N$, $S \neq \emptyset$, and so, $\vec{y} \in Core(v)$, provided $\vec{y} \in CC(v)$.

(ii) In order to prove the converse statement, suppose the coincidence $CC(v) = Core(v)$. We aim to prove (7.1.5). Let $S \subseteq N$, $S \neq \emptyset$. We distinguish two cases.

Case 1. Assume $v(N) - \vec{M}^v(N\backslash S) < \vec{m}^v(S)$. We prove $v(S) \leq \vec{m}^v(S)$. For that purpose, construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in R^N$ such that $y_i = m_i^v$ for all $i \in S$ and $y_i = m_i^v + \frac{v(N) - \vec{m}^v(N)}{(\vec{M}^v - \vec{m}^v)(N\backslash S)} \cdot (M_i^v - m_i^v)$ for all $i \in N\backslash S$. Then $m_i^v \leq y_i \leq M_i^v$ for all $i \in N\backslash S$ due to our assumption. So, $\vec{y} \in CC(v)$ and so, $\vec{y} \in Core(v)$, Thus, $\vec{y}(S) \geq v(S)$ or equivalently, $\vec{m}^v(S) \geq v(S)$.

Case 2. Assume $v(N) - \vec{M}^v(N\backslash S) \geq \vec{m}^v(S)$. We prove $v(S) \leq v(N) - \vec{M}^v(N\backslash S)$. We distinguish two subcases. Put $g^v(N) = \vec{M}^v(N) - v(N)$.

Subcase 1. Suppose there exists at least a player $i \in S$ with $M_i^v - m_i^v \geq g^v(N)$. Construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in R^N$ such that $y_i = M_i^v - g^v(N)$ and $y_j = M_j^v$ for all $j \in N\backslash\{i\}$. Then $m_j^v \leq y_j \leq M_j^v$ for all $j \in N$. So, $\vec{y} \in CC(v)$ and so, $\vec{y} \in Core(v)$. Thus, $\vec{y}(S) \geq v(S)$ or equivalently, $v(S) \leq \vec{M}^v(S) - g^v(N) = v(N) - \vec{M}^v(N\backslash S)$.

Subcase 2. Suppose $M_i^v - m_i^v < g^v(N)$ for all $i \in S$. Without loss of generality, write $S = \{i_1, i_2, \ldots, i_s\}$ such that $M_{i_1}^v - m_{i_1}^v \leq M_{i_2}^v - m_{i_2}^v \leq \ldots\ldots \leq M_{i_s}^v - m_{i_s}^v$. Then there exists $2 \leq t \leq s$ such that

$$\sum_{k=1}^{t-1}\left[M_{i_k}^v - m_{i_k}^v\right] < g^v(N) \quad \text{and} \quad \sum_{k=1}^{t}\left[M_{i_k}^v - m_{i_k}^v\right] \geq g^v(N).$$

Construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in R^N$ such that $y_i = M_i^v$ for all $i \in N\backslash S$, and $y_{i_k} = m_{i_k}^v$ for all $i_k \in S$, $k < t$ and $y_{i_k} = M_{i_k}^v$ for all $i_k \in S$, $k > t$, and $y_{i_t} = M_{i_t}^v + \sum_{k=1}^{t-1}\left[M_{i_k}^v - m_{i_k}^v\right] - g^v(N)$. Then $m_j^v \leq y_j \leq M_j^v$ for all $j \in N$. So, $\vec{y} \in CC(v)$ and so, $\vec{y} \in Core(v)$. Thus, $\vec{y}(S) \geq v(S)$ or equivalently, $v(S) \leq \vec{M}^v(S) - g^v(N) = v(N) - \vec{M}^v(N\backslash S)$. This completes the alternative proof. □

Remark 7.1. With every TU game $v : \mathcal{P}(N) \to R$, there is associated its *gap function* $g^v : \mathcal{P}(N) \to R$ defined by $g^v(S) = \vec{M}^v(S) - v(S)$ for all $S \subseteq N$, where $g^v(\emptyset) = 0$. An adapted version of (7.1.5) is well-known as the so-called 1-convexity constraint as follows:

$$v(S) \leq v(N) - \vec{M}^v(N \backslash S) \quad \text{or equivalently,} \quad g^v(N) \leq g^v(S) \quad \text{for all } S \subseteq N,\ S \neq \emptyset \tag{7.1.6}$$

In words, the TU game v is said to be 1-convex if its corresponding (non-negative) gap function g^v attains its minimum at the grand coalition. Clearly, the class of compromise stable TU games contains the subclass of 1-convex n-person games [21], [14], as well as the 2-convex n-person games [21], [16], and the big boss and clan games [46], [50], [3].

Further, from (7.1.3), we deduce that $M_i^v - m_i^v = \min\left[g^v(S) \mid \quad S \subseteq N, \quad i \in S\right]$ for all $i \in N$. Thus, $m_i^v \leq M_i^v$ if and only if $g^v(S) \geq 0$ for all $S \subseteq N$ with $i \in S$. Particularly, $\vec{m}^v \leq \vec{M}^v$ if and only if $g^v(S) \geq 0$ for all $S \subseteq N$, $S \neq \emptyset$. Throughout the next section we tacitly assume a non-negative gap function.

7.2 The indirect function as a tool for the determination of the Nucleolus of compromise stable TU games

Theorem 7.2. *Let the TU game $v : \mathcal{P}(N) \to R$ be compromise stable. Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:*

(i) $\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k \in N} y_k\right]$ *for all $\vec{y} = (y_k)_{k \in N} \in R^N$ with $\vec{m}^v \leq \vec{y} \leq \vec{M}^v$.*

(ii) For all $\vec{y} = (y_k)_{k \in N} \in R^N$ such that there exist unique $i, j \in N$ with

$y_i < m_i^v$, $y_j > M_j^v$, *and* $m_k^v \leq y_k \leq M_k^v$ *for all* $k \in N\backslash\{i,j\}$,

$$\pi^v(\vec{y}) = \max\left[m_i^v - y_i, \quad v(N\backslash\{j\}) - \sum_{k\in N\backslash\{j\}} y_k\right] \tag{7.2.1}$$

$$= \max\left[m_i^v - y_i, \quad v(N) - \sum_{k\in N} y_k + y_j - M_j^v\right] \tag{7.2.2}$$

(iii) With any efficient payoff vector $\vec{x} = (x_k)_{k\in N} \in R^N$ *satisfying* $\vec{m}^v \leq \vec{x} \leq \vec{M}^v$, *and any pair* $i, j \in N$ *of players, and any transfer* $\delta \geq 0$ *from* i *to* j, *there is associated the adapted payoff vector* $\vec{x}^{ij\delta} = (x_k^{ij\delta})_{k\in N} \in R^N$ *given by* $x_i^{ij\delta} = x_i - \delta$, $x_j^{ij\delta} = x_j + \delta$, *and* $x_k^{ij\delta} = x_k$ *for all* $k \in N\backslash\{i,j\}$. *Then, for* $\delta \geq 0$ *sufficiently large, it holds*

$$\pi^v(\vec{x}^{ij\delta}) = \delta + \max\left[m_i^v - x_i, \quad x_j - M_j^v\right] \qquad \textit{for all } i, j \in N,\ i \neq j. \tag{7.2.3}$$

(iv) For $\delta \geq 0$ *sufficiently large, the pairwise equilibrium condition* $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ *is equivalent to*

$$\min\left[x_i - m_i^v, \quad M_j^v - x_j\right] = \min\left[x_j - m_j^v, \quad M_i^v - x_i\right] \qquad \textit{for all } i, j \in N,\ i \neq j. \tag{7.2.4}$$

Proof. (i) From Theorem 7.1 we derive that for every vector $\vec{y} \in R^N$ with $\vec{m}^v \leq \vec{y} \leq \vec{M}^v$ and every coalition $S \subseteq N$, $S \neq N$, $S \neq \emptyset$,

$$v(S) - \vec{y}(S) \leq \max\left[(\vec{m}^v - \vec{y})(S), \quad v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N\backslash S)\right]$$

$$\leq \max\left[0, \quad v(N) - \vec{y}(N)\right] \qquad \text{and so,}$$

$$\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \vec{y}(N)\right] \qquad \text{for all } \vec{m}^v \leq \vec{y} \leq \vec{M}^v.$$

This completes the proof of part (i).
In order to prove part (ii), let $\vec{y} = (y_k)_{k\in N} \in R^N$ be such that there exist

$i, j \in N$ with $y_i < m_i^v$, $y_j > M_j^v$, and $m_k^v \leq y_k \leq M_k^v$ for all $k \in N \backslash \{i, j\}$. In order to study the excesses $e^v(S, \vec{y})$, $S \subseteq N$, $S \neq N$, $S \neq \emptyset$, we distinguish three cases.

Case 1. Assume $\vec{m}^v(S) \leq v(N) - \vec{M}^v(N \backslash S)$. Then it holds $v(S) \leq v(N) - \vec{M}^v(N \backslash S)$ and so,

$$\begin{aligned} v(S) - \vec{y}(S) &\leq v(N) - \vec{M}^v(N \backslash S) - \vec{y}(S) = v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N \backslash S) \\ &\leq v(N) - \vec{y}(N) + (y_j - M_j^v) = v(N \backslash \{j\}) - \vec{y}(N \backslash \{j\}) \qquad (7.2.5) \end{aligned}$$

By (7.2.5), $e^v(S, \vec{y}) \leq e^v(N \backslash \{j\}, \vec{y})$ for all $S \subseteq N$ with $\vec{m}^v(S) \leq v(N) - \vec{M}^v(N \backslash S)$.

Case 2. Assume $\vec{m}^v(S) > v(N) - \vec{M}^v(N \backslash S)$. Then it holds $v(S) \leq \vec{m}^v(S)$. We distinguish two subcases.

Subcase 1. Assume $i \notin S$. Then we derive $v(S) - \vec{y}(S) \leq (\vec{m}^v - \vec{y})(S) \leq 0$.

Subcase 2. Assume $i \in S$. Then we derive $v(S) - \vec{y}(S) \leq (\vec{m}^v - \vec{y})(S) \leq m_i^v - y_i$.

In summary, $e^v(S, \vec{y}) \leq m_i^v - y_i$ for all $S \subseteq N$ with $\vec{m}^v(S) > v(N) - \vec{M}^v(N \backslash S)$. Particularly, $e^v(\{i\}, \vec{y}) = v(\{i\}) - y_i \leq m_i^v - y_i$. Notice that $m_i^v \geq v(N) - \vec{M}^v(N \backslash \{i\})$ because of $M_i^v - m_i^v \leq g^v(N)$. Due to the forthcoming remark 7.2, we claim, without loss of generality, that $m_i^v - y_i$ equals the excess $e^v(\{i\}, \vec{y})$. Hence, (7.2.1) holds, or equivalently, (7.2.2). As a direct consequence, (7.2.3)–(7.2.4) hold. □

Remark 7.2. Whenever $m_i^v \neq v(\{i\})$, the latter proof of Theorem 7.2 has to be adapted by means of a slight change of the worth of player i without changing the Core and Nucleolus concept. Formally, with a TU game $v : \mathcal{P}(N) \to R$ and a fixed player $i \in N$, there is associated the TU game $w : \mathcal{P}(N) \to R$ given by $w(\{i\}) = m_i^v$ and $w(S) = v(S)$ for all $S \subseteq N$, $S \neq \{i\}$. Clearly, $\vec{M}^w = \vec{M}^v$ as well as $\vec{m}^w = \vec{m}^v$. Moreover, by (7.1.2), both games possess the same Core because $m_i^v \geq v(\{i\})$ as well as $\vec{m}^v$ represents a lower bound for $Core(v)$. Consequently, the intersection of the Core with the Prekernel is the same for both games [43] and for the classes under consideration, it follows from the uniqueness part that both games have the same Nucleolus. Finally, by Theorem 7.1, if the game v is compromise stable, then the game w is compromise stable too: $w(\{i\}) = m_i^v = m_i^w \leq \max\Big[m_i^w, \quad w(N) -$

$\vec{M}^w(N\backslash\{i\})\Big]$, and for all $S \subseteq N$, $S \neq \emptyset$, $S \neq \{i\}$,

$$w(S) = v(S) \leq \max\Big[\vec{m}^v(N), \quad v(N) - \vec{M}^v(N\backslash S)\Big]$$
$$= \max\Big[\vec{m}^w(N), \quad w(N) - \vec{M}^w(N\backslash S)\Big]$$

Without going into details [45], [26], we state that the pairwise equilibrium conditions $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for all pairs $i, j \in N$ of players and for $\delta \geq 0$ sufficiently large, fully determine the so-called *Pre-kernel* of the TU game v [43]. As a matter of fact, the set of efficient solutions of the non-linear system of equations (7.2.4) is unique and it is a so-called constrained equal award rule of the parametric form

$$x_i = m_i^v + \max\Big[M_i^v - m_i^v - \lambda, \quad \frac{M_i^v - m_i^v}{2}\Big] \qquad \text{for all } i \in N, \qquad (7.2.6)$$

where the parameter $\lambda \in R$ is determined by the efficiency constraint $\vec{x}(N) = v(N)$. This unique solution within the Pre-kernel is well-known as the Nucleolus of the TU game v. In [41], the approach to determine the Nucleolus of compromise stable TU games is totally different and strongly based on the study of (convex) bankruptcy games [41] (Theorem 4.2, pages 497-498).

7.3 The indirect function and Nucleolus of clan TU games

Definition 7.4. [50], [46], [3]
An n-person TU game $v : \mathcal{P}(N) \to R$ is said to be a *clan game* if $M_i^v \geq v(\{i\})$ for all $i \in N$ and there exists a coalition $T \subseteq N$, called the *clan*, such that $v(S) = 0$ whenever $T \not\subseteq S$ and

$$v(S) \leq v(N) - \vec{M}^v(N\backslash S) \qquad \text{for all } S \subseteq N,\ S \neq \emptyset, \text{ with } T \subseteq S \quad (7.3.1)$$

A clan game v with an empty clan reduces to an 1-convex game, provided $g^v(N) \geq 0$. A clan game with the clan to be a singleton is known as a big boss game.

Throughout this section we suppose that the clan T consists of at least two players.

Theorem 7.3. *Let the n-person TU game $v : \mathcal{P}(N) \to R$ be a clan game, Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:*

(i) $\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k\in N} y_k\right]$ *for all* $\vec{y} = (y_k)_{k\in N} \in R^N$ *with* $y_i \geq 0$ *for all* $i \in N$ *and* $y_i \leq M_i^v$ *for all* $i \in N\backslash T$.

(ii) $\pi^v(\vec{y}) = \max\left[0, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\right] = \max\left[0, \quad v(N) - \sum_{k\in N} y_k + y_\ell - M_\ell^v\right]$ *for all* $\vec{y} = (y_k)_{k\in N} \in R^N$ *such that there exists a unique* $\ell \in N\backslash T$ *with* $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ *for all* $i \in N\backslash T$, $i \neq \ell$, *and* $y_i \geq 0$ *for all* $i \in N$.

(iii) $\pi^v(\vec{y}) = \max\left[-y_\ell, \quad v(N) - \sum_{k\in N} y_k\right]$ *for all* $\vec{y} = (y_k)_{k\in N} \in R^N$ *such that there exists a unique* $\ell \in N$ *with* $y_\ell < 0$, $y_i \geq 0$ *for all* $i \in N\backslash\{\ell\}$, *and* $y_i \leq M_i^v$ *for all* $i \in N\backslash T$.

(iv) $\pi^v(\vec{y}) = \max\left[-y_j, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\right] = \max\left[-y_j, \quad v(N) - \sum_{k\in N} y_k + y_\ell - M_\ell^v\right]$ *for all* $\vec{y} = (y_k)_{k\in N} \in R^N$ *such that there exist unique* $j \in N$, $\ell \in N\backslash T$ *with* $y_j < 0$, $y_i \geq 0$ *for all* $i \in N\backslash\{j\}$, *and* $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ *for all* $i \in N\backslash T$, $i \neq \ell$.

Proof. Let $\vec{y} = (y_k)_{k\in N} \in R^N$.
(i) Suppose that $y_i \geq 0$ for all $i \in N$ and $y_i \leq M_i^v$ for all $i \in N\backslash T$. We distinguish two types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $T \subseteq S$, then we derive from (7.3.1),

$$\begin{aligned} & v(S) - \vec{y}(S) \leq v(N) - \vec{M}^v(N\backslash S) - \vec{y}(S) \\ = \quad & v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N\backslash S) \leq v(N) - \vec{y}(N). \end{aligned} \tag{7.3.2}$$

This proves part (i). In order to prove part (ii), suppose that there exists a unique $\ell \in N\backslash T$ with $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N\backslash T$, $i \neq \ell$, and $y_i \geq 0$ for all $i \in N$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$.

In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $T \subseteq S$, together with $\ell \in S$, then $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ as shown in (7.3.2). In case $T \subseteq S$, together with $\ell \notin S$, then we derive from (7.3.1)

$$\begin{aligned} v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + y_l + \vec{y}(N\backslash(S \cup \{l\})) \\ &\leq v(S) - \vec{y}(N) + y_l + \vec{M}^v(N\backslash(S \cup \{l\})) \\ &= v(S) - \vec{y}(N) + y_\ell - M_l^v + \vec{M}^v(N\backslash S) \\ &\leq v(N) - \vec{y}(N) + y_l - M_l^v = v(N\backslash\{l\}) - \vec{y}(N\backslash\{l\}) \end{aligned} \tag{7.3.3}$$

In this setting, the indirect function π^v attains its maximum either for $S = N$, $S = N\backslash\{\ell\}$ or $S = \emptyset$, but $S = N$ cancels. The similar proof of part (iii) is omitted here.
(iv) Suppose that there exist unique $j \in N$, $\ell \in N\backslash T$ with $y_j < 0$, $y_i \geq 0$ for all $i \in N\backslash\{j\}$, and $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N\backslash T$, $i \neq \ell$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq -y_j$. In case $T \subseteq S$, the proof proceeds similar to the proof of part (ii) and is omitted too. □

Corollary 7.1. *For every n-person clan game $v : \mathcal{P}(N) \to R$, with clan T, the following three statements concerning a payoff vector $\vec{y} = (y_k)_{k \in N} \in R^N$ are equivalent.*

(i) $\vec{y} \in Core(v)$, *i.e.,* $\vec{y}(N) = v(N)$ *and* $\vec{y}(S) \geq v(S)$ *for all* $S \subseteq N$, $S \neq \emptyset$

(ii) $\vec{y}(N) = v(N)$ *and* $\pi^v(\vec{y}) = 0$

(iii) $\vec{y}(N) = v(N)$ *and* $y_i \geq 0$ *for all* $i \in N$ *and* $y_i \leq M_i^v$ *for all* $i \in N\backslash T$

Theorem 7.4. *Let the n-person TU game $v : \mathcal{P}(N) \to R$ be a clan game with clan T. For $\delta \geq 0$ sufficiently large, the pairwise equilibrium conditions $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for all pairs $i, j \in N$ of players reduce to the following system of equations:*

Case	***Pairwise equilibrium equation*** $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$
$i \in T,\ j \in T,$	$\max\begin{bmatrix}-(x_i-\delta), & 0\end{bmatrix} = \max\begin{bmatrix}-(x_j-\delta), & 0\end{bmatrix}$
$i \notin T,\ j \in T,$	$\max\begin{bmatrix}-(x_i-\delta), & 0\end{bmatrix} = \max\begin{bmatrix}-(x_j-\delta), & (x_i+\delta)-M_i^v\end{bmatrix}$
$i \notin T,\ j \notin T,$	$\max\begin{bmatrix}-(x_i-\delta), & (x_j+\delta)-M_j^v\end{bmatrix}$
	$= \max\begin{bmatrix}-(x_j-\delta), & (x_i+\delta)-M_i^v\end{bmatrix}$

Case	***Resulting pairwise equation for*** $\vec{x} = (x_k)_{k\in N} \in R^N$
$i \in T,\ j \in T,$	$x_i = x_j$
$i \notin T,\ j \in T,$	$x_i = \min\begin{bmatrix}x_j, & M_i^v - x_i\end{bmatrix}$
$i \notin T,\ j \notin T,$	$\min\begin{bmatrix}x_i, & M_j^v - x_j\end{bmatrix} = \min\begin{bmatrix}x_j, & M_i^v - x_i\end{bmatrix}$

The proof follows immediately by explicit formula for the indirect function of clan games, as presented in Theorem 7.3 (ii)–(iv).

In summary, the unique solution is a so-called constrained equal reward rule of the form $x_i = \lambda$ *for all* $i \in T$ *and* $x_i = \min\begin{bmatrix}\lambda, & \frac{b_i^v}{2}\end{bmatrix}$ *for all* $i \in N\backslash T$*, where the parameter* $\lambda \in R$ *is determined by the efficiency condition* $\vec{x}(N) = v(N)$*.*

Further, the indirect function is a helpful tool for the determination of the Nucleolus for the subclasses of big boss games as well as 1-convex and 2-convex n-person games [26], [16].

Chapter 8

Interaction between Dutch Soccer Teams and Fans: A mathematical analysis through Cooperative game theory

ABSTRACT - In this chapter, we model the interaction between soccer teams and their potential fans as a cooperative cost game based on the annual voluntary sponsorships of fans in order to validate their fan registration in a central database, inspired by the first lustrum of the Club Positioning Matrix (CPM) for professional Dutch soccer teams. The game theoretic approach aims to show that the so-called Nucleolus of the suitably chosen fan data cost game agrees with the deviations of b_i, $i \in N$, *from their average, where* b_i *represents the total budget of sponsorships of fans whose unique favorite soccer team is* i.

8.1 Club Positioning Matrix (CPM) of professional Dutch soccer

Five years CPM. The first lustrum of the (Dutch) "Eredivisie Effectenbeurs" is a fact. At the initiative of the eighteen professional teams in the Dutch soccer league called "Eredivisie", the first CPM research has been carried out October 2006 by one of the German leading research and consultancy companies in international sport business (i.e., marketing and sponsoring) called "Sport+Markt" (www.sportundmarkt.com). October 2009 the fourth CPM research involved 4.500 participants randomly selected from the whole Dutch population (with the common feature to be a fan of soccer). Since these soccer teams were not satisfied at all by the point of time October, the fifth CPM research edition has been carried out among 4.500 participants in two stages, namely August 2010 at the beginning of the soccer season and January 2011 during the soccer winter break. The end of March 2011, the CPM 2011 scores have been sent to the professional Dutch soccer teams and published exclusively in the weekly Dutch soccer magazine "Voetbal International" (www.vi.nl). [1]

The CPM 2011 scores have direct consequences for the participating soccer teams since the allocation of media (television and broadcast) money among all the soccer teams is based equally on both the annual sport results and the average CPM scores over three years. The more CPM points, the more media money. During one half of a century, the annual sport results were dominated fully by the triple PSV Eindhoven (last Dutch championships in 2000, 2001, 2003, 2005, 2006, 2007, 2008), Ajax Amsterdam (2002, 2004, 2011), and Feyenoord Rotterdam (1974, 1984, 1993, 1999), with exceptions caused by DWS in 1964, AZ'67 Alkmaar in 1981 as well as 2009, and FC Twente Enschede in 2010. The top five of the last three annual sport results is as follows (R = ranking):

[1] The CPM 2011 scores have been published in the weekly soccer magazine "Voetbal International" VI (in Dutch), Volume 46, March 30, 2011, nr. 13, pp. 116–121, co-authored by Iwan van Duren and Tom Knipping, e-mail addresses duren@vi.nl, knipping@vi.nl Two subsequent articles have been published in VI by the same authors April 6, 2011, nr. 14, pp. 116–119, and April 20, 2011, nr. 16, pp. 118–121.

R	2010–2011	Score	2009–2010	Score	2008–2009	Score
1	Ajax	73	FC Twente	86	AZ'67	80
2	FC Twente	71	Ajax	85	FC Twente	69
3	PSV	69	PSV	78	Ajax	68
4	AZ'67	59	Feyenoord	63	PSV	65
5	FC Groningen	57	AZ'67	62	SC Heerenveen	60

The annual CPM is a marketing instrument that measures the marketing value (through a professional jury of marketing specialists) as well as the imago of every professional soccer team (through the randomly selected soccer fans), which, in turn, is determined on the basis of six parts. Finally, the marketing value, the imago, and the annual sport result are put into some calculation model yielding the annual CPM scores.

The top five of the best marketing in 2011 is as follows: (1) PSV; (2) SC Heerenveen; (3) FC Twente; (4) Ajax; (5) Feyenoord. Like the fourth edition, the team with the best imago is FC Twente due to its unique national championship, its successful participation in the international Champions League as well as the European League (till the quarter finales), and its new stadium called Grolsch Veste. FC Twente's imago is the best in the subfields attraction (charm), fascination, economical success, and the second best in the subfields emotional involvement and identification. The top six of the best imago is as follows: (1) FC Twente 699; (2) Ajax 641; (3) PSV 624; (4) SC Heerenveen 558; (5) FC Groningen 539; (6) Feyenoord 476.

In summary, the CPM score of FC Twente increased drastic, Feyenoord's score decreased drastic, so that the third ranking in the CPM 2011 scores is occupied by FC Twente. Ajax and PSV remain first and second due to the CPM results of the previous years. The top seven of the last three final CPM rankings is as follows (R = ranking):

R	CPM 2011	Score	CPM 2010	Score	CPM 2009	Score
1	Ajax	2.928	Ajax	2.888	Ajax	2.791
2	PSV	2.649	PSV	2.568	PSV	2.656
3	FC Twente	2.269	Feyenoord	2.237	Feyenoord	2.255
4	Feyenoord	2.199	AZ'67	2.106	AZ'67	2.165
5	AZ'67	2.086	FC Twente	2.059	SC Heerenveen	2.104
6	SC Heerenveen	2.071	SC Heerenveen	1.943	FC Groningen	1.804
7	FC Groningen	1.780	FC Groningen	1.552	FC Twente	1.661

Concerning the fan status, the CPM 2011 top five is as follows: (1) Ajax; (2) Feyenoord; (3) PSV; (4) FC Twente; (5) AZ'67. During the first CPM research October 2006, FC Twente started with a fanstatus of 250.000 fans, nowadays its fan status has been increased up to about 1.6 million, being the double of its previous edition.

8.2 The fan database model

Given the current fan status as the model of the interaction between the professional Dutch soccer teams and their potential fans, our main goal is to apply the solution part of the mathematical field called "Cooperative game theory". The so-called "players" are the soccer teams, each of which is endowed with a set of potential fans, each of which is supposed to validate its fan registration in a central database through an annual voluntary sponsorship to be cashed to the national soccer association. This annual sponsorship is said to be voluntary since it varies from fan to fan, each fan decides by him/herself about the contribution to be small or large. No registration if the potential fan is not willing to fulfil this sponsorship. In fact, any commitment to this sponsorship guarantees certain priorities to the fan, such as priority rights to purchase tickets for additional (inter)national soccer matches with or without discount, program booklets free of charge, and so on. Notice that any fan is allowed to be registrated (in a central database run by the national soccer association) for a number of distinct soccer teams (not necessarily one team), while contributing the annual voluntary sponsorship once (at the beginning of the soccer season). The next table surveys the essential notions.

Symbol	**Interpretation**	**Symbol**	**Interpretation**
N	set of soccer teams	$i \in N$	i is called player
D_i	set of fans of soccer team i	$j \in D_i$	fan j of soccer team i
$D = \cup_{i \in N} D_i$	set of all fans	$j \in D$	fan j

Symbol	**Interpretation**
$S, S \subseteq N$	subset of soccer teams
$i \in S$	soccer team i of coalition S
$D_S = \cup_{i \in S} D_i$	subset of fans of at least one soccer team of S
$N_j = \{i \in N \mid j \in D_i\}$	set of soccer teams of which j is a fan equivalence: $i \in N_j$ iff $j \in D_i$
$s_j > 0$	annual voluntary sponsorship of fan j in order to validate the fan registration
$\sum_{j \in D} s_j$	total sponsorship of all fans
$\sum_{j \in D_S} s_j$	coalitional sponsorship a of fans of at least one soccer team of S

Symbol	**Interpretation**
$\sum_{j\in D} s_j - \sum_{j\in D_S} s_j$	coalitional loss (shortage) of sponsorship of soccer teams of coalition S
$c(S) = \sum_{j\in D} s_j - \sum_{j\in D_S} s_j$	coalitional cost equals the loss of coalitional sponsorship versus total sponsorship
$c(N) = 0$	motivation for cooperation to form N due to minimization of shortages of sponsorship

In summary, the fan database of professional Dutch soccer teams may be modeled as the triple $(N, (D_i)_{i\in N}, (s_j)_{j\in D})$ such that the player set N consists of the soccer teams, the set D_i consists of fans of soccer team i, and $s_j > 0$ represents the annual voluntary sponsorship of fan j. In fact, these sponsorships are combined to construct the following cost allocation $\vec{y} = (y_i)_{i\in N} \in R^N$.
Consider the budget B as the sum of sponsorships of fans with a unique (unspecified) favourite soccer team (that is, $j \in D$ with $|N_j| = 1$). Factorize this budget in accordance with the appearance of the unique soccer team involved, that is $B = \sum_{i\in N} b_i$, with the understanding that $b_i = 0$ if there are no fans $j \in D$ with unique favourite soccer team i. Finally, with reference to these factorizations, determine the deviations with respect to their average. In summary, charge to soccer team i the cost allocation amounting $y_i = \frac{B}{|N|} - b_i$ for all $i \in N$. In words, reward to soccer team i the negative amount $-b_i$, and charge the budget B equally among all the soccer teams. In particular, a soccer team i receives a reward (instead of a cost charge) if and only if the total sponsorship b_i exceeds the average $\frac{B}{|N|}$ of the budget. The larger b_i, the larger the reward to soccer team i. That is, soccer teams benefit from fans who are willing to contribute a large sponsorship. The second table surveys the essential notions in the setting of cost allocations. Let $|X|$ denote the cardinality of any finite set X.

Formula	Interpretation
$B = \sum_{\substack{j \in D,\\ \|N_j\|=1}} s_j$	the total sponsorship of fans with a unique unspecified favourite soccer team
$b_i = \sum_{\substack{j \in D,\\ N_j=\{i\}}} s_j$	the total sponsorship of fans with the unique specified favourite soccer team i
$y_i = \frac{B}{\|N\|} - b_i$	cost allocation charged to soccer team i

8.3 The game theoretic model and the Nucleolus

Our main goal is to support the cost allocation $(y_i)_{i \in N}$ from the viewpoint of Cooperative game theory as the so-called Nucleolus [53] of the suitably chosen fan data cost game $\langle N, c\rangle$ defined by $c(\emptyset) = 0$ and

$$c(S) = \sum_{j \in D} s_j - \sum_{j \in D_S} s_j = \sum_{j \in D \backslash D_S} s_j \qquad \text{for all } S \subseteq N,\ S \neq \emptyset \tag{8.3.1}$$

For instance, for any $i \in N$, the fan data cost

$$c(N \backslash \{i\}) = \sum_{j \in D \backslash D_{N \backslash \{i\}}} s_j = \sum_{\substack{j \in D,\\ N_j=\{i\}}} s_j = b_i \tag{8.3.2}$$

Here the second equality is due to the following equivalences (for fixed $i \in N$):

$$\begin{aligned}
& N_j = \{i\} \\
\Leftrightarrow\ & i \in N_j \text{ and } k \notin N_j \text{ for all } k \neq i \\
\Leftrightarrow\ & j \in D_i \text{ and } j \notin D_k \text{ for all } k \neq i \\
\Leftrightarrow\ & j \in D_i \text{ and } j \notin D_{N \backslash \{i\}} \\
\Leftrightarrow\ & j \in D \backslash D_{N \backslash \{i\}}
\end{aligned}$$

Note that the soccer teams are willing to cooperate (to share the fan data information of the central database) in order to solve the minimization problem

of shortages of sponsorships in that $c(N) = 0$ reflecting the formation of the grand coalition.
Our proof technique involves the notion of excess $e(T, \vec{x}) = c(T) - \sum_{i \in T} x_i$, where $T \subseteq N$, $T \neq N$, $T \neq \emptyset$, and $\vec{x} = (x_i)_{i \in N} \in R^N$. It turns out that the level of the smallest excesses with respect to our cost allocation $\vec{y} = (y_i)_{i \in N}$ is composed of all the $(|N| - 1)$-person coalitions $N \backslash \{i\}$, $i \in N$, due to the following fundamental property of the fan data cost game $\langle N, c \rangle$ respectively the sponsorships s_j, $j \in D$, of fans: for all $T \subseteq N$, $T \neq N$, $T \neq \emptyset$,

$$c(T) \geq \sum_{i \in N \backslash T} b_i \qquad \text{or equivalently, by (8.3.1),} \tag{8.3.3}$$

$$\sum_{j \in D \backslash D_T} s_j \geq \sum_{i \in N \backslash T} \sum_{\substack{j \in D, \\ N_j = \{i\}}} s_j \tag{8.3.4}$$

The validity of (8.3.4) follows immediately from the inclusion $\{j \in D \mid N_j = \{i\}, \quad i \in N \backslash T\} \subseteq D \backslash D_T$. Consequently, by (8.3.3), the excess $e(T, \vec{y})$ with respect to our cost allocation $\vec{y}$ satisfies

$$\begin{aligned} e(T, \vec{y}) &= c(T) - \sum_{i \in T} y_i = c(T) + \sum_{i \in T} b_i - \frac{B}{|N|} \cdot |T| \\ &\geq \sum_{i \in N} b_i - \frac{B}{|N|} \cdot |T| = B - \frac{B}{|N|} \cdot |T| \geq \frac{B}{|N|} \end{aligned}$$

whereas, for all $i \in N$,

$$e(N \backslash \{i\}, \vec{y}) = c(N \backslash \{i\}) - \sum_{k \in N} y_k + y_i = c(N \backslash \{i\}) - 0 + \frac{B}{|N|} - b_i = \frac{B}{|N|}$$

where the latter equality is due to (8.3.2).

Hence, all $(|N| - 1)$-person coalitions have the smallest excess among non-trivial coalitions with respect to our cost allocation $\vec{y}$ and according to Kohlberg's criterion [34] , this suffices to conclude that our cost allocation $\vec{y}$ agrees with the Nucleolus of the fan data cost game $\langle N, c \rangle$ of the form (8.3.1).

8.4 The empty Core of the sponsorship game

In the setting of the division problem of the total budget B among the soccer teams, it is natural to study the sponsorship game $\langle N, v\rangle$ defined by

$$v(S) = \sum_{j \in D_S} s_j \qquad \text{for all } S \subseteq N,\ S \neq \emptyset \tag{8.4.1}$$

Clearly, straight from its definition, $v(N) \geq B$. Unfortunately, this sponsorship game has the drawback that its so-called "Core" is empty. For that purpose, consider the set of "reasonable" payoff vectors of the game consisting of efficient payoff vectors $\vec{x} = (x_i)_{i \in N} \in R^N$ satisfying $\sum_{i \in N} x_i = v(N)$ as well as lower and upper bounds for the individual payoffs such that $v(\{i\}) \leq x_i \leq v(N) - v(N \backslash \{i\})$ for all $i \in N$. In the context of the sponsorship game $\langle N, v\rangle$ of the form (8.4.1), it holds for all $i \in N$

$$v(N) - v(N \backslash \{i\}) = \sum_{j \in D} s_j - \sum_{j \in D_{N \backslash \{i\}}} s_j = \sum_{j \in D \backslash D_{N \backslash \{i\}}} s_j = c(N \backslash \{i\}) = b_i$$

where the last equation is due to 8.3.2. Thus any reasonable payoff vector $\vec{x}$ satisfies $x_i \leq b_i$ for all $i \in N$ and consequently, by summing up, $v(N) \leq B$. This contradicts the earlier observation $v(N) \geq B$. So, reasonable payoff vectors do not exist, and hence, the Core of the sponsorship game is empty too (as a subset).

8.5 Concluding Remarks

Inspired by the first lustrum of the Club Positioning Matrix (CPM) for professional Dutch soccer teams, we model the interaction between soccer teams and their potential fans as a cooperative cost game based on the annual voluntary sponsorships of fans in order to validate their fan registration in a central database. We introduce a natural cost allocation to the soccer teams, based in a natural manner on the sponsorships of fans. The game theoretic approach is twofold. On the one hand, an appropriate cost game called "fan data cost game" is developed and on the other, it is shown that the former natural cost allocation agrees with the solution concept called "Nucleolus" of the fan data cost game.

Chapter 9

Data Cost Games as an Application of 1-Concavity in Cooperative game theory

ABSTRACT - In this chapter, The main goal is to reveal the 1*-concavity property for a subclass of cost games called Data Cost Games. Two significantly different proofs are treated. The motivation for the study of the* 1*-concavity property are the appealing theoretical results for both the Core and the nucleolus, in particular their geometrical characterization as well as their additivity property. The characteristic cost function of the original Data Cost Game assigns to every coalition the additive cost of reproducing the data the coalition does not own. The underlying data and cost sharing situation is composed of three components, namely the player set, the collection of data sets for individuals, and the additive cost function on the whole data set. The first proof of* 1*-concavity is direct, but robust to a suitable generalization of the characteristic cost function. The second proof of* 1*-concavity is based on a suitably chosen decomposition of the data cost game which invites to a close comparison between the nucleolus and the Shapley cost allocations.*

9.1 The Data Sharing Situation and the Data Cost Game

This chapter broadens the game theoretic approach to the data sharing situation initiated by Pierre Dehez and Daniela Tellone [7]. The origin of their mathematical study is the *data and cost sharing problem* faced by the European chemical industry. Following the regulation imposed by the European Commission under the acronym "REACH" (Registration, Evaluation, Authorization and restriction of CHemical substances), manufacturers and importers are required to collect safety information on the properties of their chemical substances. There are about 30,000 substances and an average of 100 parameters for each substance. Chemical firms are required to register the information in a central database run by the European Chemicals Agency (ECHA). By 2018, this regulation program REACH requires submission of a detailed analysis of the chemical substances produced or imported. Chemical firms are encouraged to cooperate by sharing the data they have collected over the past. To implement this data sharing problem, a compensation mechanism is needed.

This data sharing problem can be specified as follows. A finite group of firms agrees to undertake a joint venture that requires the combination of various complementary inputs held by some of them. These inputs are non-rival but excludable goods, i.e., public goods with exclusion such as knowledge, data or information, patents or copyrights (the consumption of which by individuals can be controlled, measured, and subjected to payment or other contractual limitations). In what follows we use the common term *data* to cover generically these goods. Each firm owns a subset of data. No a priori restrictions are imposed on the individual data sets. In addition, with each type of data there is associated its *replacement cost*, e.g., the present cost of duplicating the data (or the cost of developing alternative technologies). Because these public goods are already available, their costs are sunk. In summary, the data sharing situation involves a finite group of agents, data sets owned by individual agents, as well as a discrete list of costs of data.

In the setting of cooperative attitudes by chemical firms, the main question arises how to compensate the firms for the data they contribute to share. The design of a compensation mechanism, however, is fully equivalent to the selection among existing solution concepts in the mathematical field called Cooperative game theory. In fact, the solution part of Cooperative game theory aims at solving any allocation problem by proposing rules based on certain fairness properties. For that purpose, the data and cost sharing situation needs to be interpreted as a mathematical model called a *cooperative game* by specifying its fundamental characteristic cost function. We adopt Dehez and Tellone's game theoretic model in which the cost associated to any non-empty group of agents is simply the sum of costs of the missing data, i.e., the total cost of data the group does not own. In this framework, no cost are charged to the whole group of agents. The so-called *data cost games* are therefore compensation games to which standard cost allocation rules can be applied, such as the Shapley value [52], [56], the nucleolus [53], the Core and so on. The determination of these game theoretic solution concepts may be strongly simplified whenever the underlying characteristic cost function satisfies, by chance, one or another appealing property. The main purpose of this chapter is to establish the so-called 1-*concavity* property for the class of data cost games, which property has not yet been revealed. Two significantly different proofs are treated, each of which exploited in its own interest. The impact of the 1-concavity property is fundamental for the uniform determination of both solution concepts the Core and the nucleolus [21].

Definition 9.1. [7]

(i) A *data and cost sharing situation* is given by the 3-tuple $\mathcal{DC} = (N, \mathcal{D}, \mathcal{C})$ where N is the finite set of *agents*, $\mathcal{D} = (D_i)_{i \in N}$ a collection of sets $D_i \subseteq D$, $i \in N$, of *data*, and $\mathcal{C} = (c_j)_{j \in D}$ a collection of costs of data. So, $D = \cup_{i \in N} D_i$ denotes the whole data set.

(ii) Given the set N of agents, let $\mathcal{P}(N) = \{S \mid S \subseteq N\}$ denote the powerset of N. For every *coalition* $S \subseteq N$, $S \neq \emptyset$, let $D_S = \cup_{i \in S} D_i$ denote the *data set of* S. For every subset $A \subseteq D$ of data, let $c(A) = \sum_{j \in A} c_j$ denote its *additive cost*, whereas $c(\emptyset) = 0$.

(iii) With every data and cost sharing situation $\mathcal{DC} = (N, \mathcal{D}, \mathcal{C})$, there is associated the *Data Cost Game* $\langle N, C_{\mathcal{DC}} \rangle$, of which the *characteristic cost function* $C_{\mathcal{DC}} : \mathcal{P}(N) \to R$ is given by $C_{\mathcal{DC}}(\emptyset) = 0$ and for all

$S \subseteq N$, $S \neq \emptyset$,

$$C_{\mathcal{DC}}(S) = \sum_{j \in D \backslash D_S} c_j \qquad \text{Shortly,} \quad C_{\mathcal{DC}}(S) = c(D \backslash D_S) = c(D) - c(D_S) \tag{9.1.1}$$

By (9.1.1), the so-called *data cost* $C_{\mathcal{DC}}(S)$ of coalition S equals the *additive cost* of duplicating the missing data, i.e., costs of data the coalition does not own. Without loss of generality, it is tacitly supposed that there exist no overall missing data, that is $D = D_N$; otherwise the data cost of every non-empty coalition S would increase with the same cost amounting $c(D \backslash D_N) = c(D) - c(D_N)$. In our framework, no data cost are charged to the whole set of agents, i.e., $C_{\mathcal{DC}}(N) = 0$. Obviously, every data cost game $\langle N, C_{\mathcal{DC}} \rangle$ satisfies both the *(decreasing) monotonicity* (i.e., $C_{\mathcal{DC}}(S) \geq C_{\mathcal{DC}}(T)$ for all $S \subseteq T \subseteq N$, $S \neq \emptyset$, due to $D_S \subseteq D_T$) and *subadditivity* as well (i.e., $C_{\mathcal{DC}}(S \cup T) \leq C_{\mathcal{DC}}(S) + C_{\mathcal{DC}}(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$).

Definition 9.2. [22], [21], [23] A *cooperative cost game* $\langle N, C \rangle$ with player set N is said to satisfy the 1-*concavity* property if its characteristic cost function $C : \mathcal{P}(N) \to R$ satisfies

$$C(N) \leq C(S) + \sum_{i \in N \backslash S} \Delta_i(N, C) \quad \text{for all } S \subseteq N,\ S \neq N,\ S \neq \emptyset, \text{ and} \tag{9.1.2}$$

$$C(N) \geq \sum_{i \in N} \Delta_i(N, C) \quad \text{where} \quad \Delta_i(N, C) = C(N) - C(N \backslash \{i\}) \quad \text{for all } i \in N. \tag{9.1.3}$$

Condition (9.1.2) requires that the cost $C(N)$ of the formation of the grand coalition N can be covered by any coalitional cost $C(S)$ together with the marginal costs $\Delta_i(N, C)$, $i \in N \backslash S$, of all the complementary players. According to condition (9.1.3), all these marginal costs are weakly insufficient to cover the overall cost $C(N)$. In the framework of data cost games, the latter condition (9.1.3) holds trivially due to the compensation assumption $C_{\mathcal{DC}}(N) = 0$. The next two sections are devoted to two significantly different proofs of the 1-concavity property for data cost games. Each of the two proofs has its own peculiarities and generalizations.

9.2 1-Concavity of the Data Cost Game: 1st proof

Theorem 9.1. *Every data cost game* $\langle N, C_{\mathcal{D}C}\rangle$ *of the form (9.1.1) satisfies* 1*-concavity.*

Proof. Let $\langle N, C_{\mathcal{D}C}\rangle$ be a data cost game. Fix coalition $S \subseteq N$, $S \neq N$, $S \neq \emptyset$. We establish the 1-concavity inequality (9.1.2) applied to $\langle N, C_{\mathcal{D}C}\rangle$. Because of the compensation assumption $C_{\mathcal{D}C}(N) = 0$, the condition (9.1.2) reduces to

$$C_{\mathcal{D}C}(S) \quad \geq \quad \sum_{i \in N\backslash S} C_{\mathcal{D}C}(N\backslash\{i\}) \qquad \text{or equivalently, by (9.1.1),}$$

$$c(D) - c(D_S) \quad \geq \quad \sum_{i \in N\backslash S} \Big[c(D) - c(D_{N\backslash\{i\}})\Big] \tag{9.2.1}$$

Write $N\backslash S = \{i_1, i_2, \ldots, i_{n-s}\}$ where $n - s$ denotes the cardinality of $N\backslash S$. Define, for every $0 \leq k \leq n - s$, the data set $A_{i_k} = D_S \cup_{\ell=1}^{k} D_{i_\ell}$ where $A_{i_0} = D_S$, $A_{i_{n-s}} = D_N = D$. In this setting, using a telescoping sum, (9.2.1) is equivalent to

$$\sum_{k=1}^{n-s} \Big[c(A_{i_k}) - c(A_{i_{k-1}})\Big] \geq \sum_{k=1}^{n-s} \Big[c(D) - c(D_{N\backslash\{i_k\}})\Big] \tag{9.2.2}$$

In view of (9.2.2), it suffices to show the following: for all $1 \leq k \leq n - s$

$$c(A_{i_k}) - c(A_{i_{k-1}}) \quad \geq \quad c(D) - c(D_{N\backslash\{i_k\}}) \quad \text{or equivalently,} \tag{9.2.3}$$

$$\sum_{j \in A_{i_k}\backslash A_{i_{k-1}}} c_j \quad \geq \quad \sum_{j \in D\backslash D_{N\backslash\{i_k\}}} c_j \tag{9.2.4}$$

In view of (9.2.4), in turn, it suffices to show the inclusion $D\backslash D_{N\backslash\{i_k\}} \subseteq A_{i_k}\backslash A_{i_{k-1}}$ for all $1 \leq k \leq n - s$. Finally, note that $j \in D\backslash D_{N\backslash\{i_k\}}$ means $j \in D_{i_k}$, but $j \notin D_{i_\ell}$ for all $\ell \neq k$. □

Notice that the equivalence of (9.2.3) and (9.2.4) in the proof of Theorem 9.1 is due to the additive cost assumption in that $c(A) = \sum_{j \in A} c_j$ for any data subset $A \subseteq D$. We claim that the 1-concavity property is still valid when the characteristic cost function $C_{\mathcal{DC}} : \mathcal{P}(N) \to R$ is of the following generalized form: there exists a real number $\beta \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ such that

$$C_{\mathcal{DC}}(S) = \Big[\sum_{j\in D} c_j\Big]^{\beta} - \Big[\sum_{j\in D_S} c_j\Big]^{\beta} \quad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{9.2.5}$$

By (9.2.5), the data cost of coalition S equals the *surplus of costs of data* that the coalition does not own, where the surplus is measured by some concave utility function $u(x)$ of the form $x^{\frac{1}{\alpha}}$ such that α is any natural number (the case $\alpha = 1$ agrees with the additive cost setting).

Theorem 9.2. *Every generalized data cost game $\langle N, C_{\mathcal{DC}}\rangle$ of the form (9.2.5) satisfies the* 1*-concavity property.*

Proof. It suffices to prove the equivalent version of (9.2.3) as follows: for all $1 \le k \le n - s$

$$\Big[\sum_{j\in A_{i_k}} c_j\Big]^{\beta} - \Big[\sum_{j\in A_{i_{k-1}}} c_j\Big]^{\beta} \ge \Big[\sum_{j\in D} c_j\Big]^{\beta} - \Big[\sum_{j\in D_{N\setminus\{i_k\}}} c_j\Big]^{\beta} \tag{9.2.6}$$

Write $\alpha = \frac{1}{\beta}$. We make use of the fundamental calculus relationship:

$$x - y = \Big[x^{\beta} - y^{\beta}\Big] \cdot \Big[\sum_{p=0}^{\alpha-1} (x^{\beta})^{\alpha-1-p} \cdot (y^{\beta})^{p}\Big] \qquad \text{for all } x, y \in R.$$

Fix $1 \le k \le n - s$. This fundamental calculus relationship applied to the validity of (9.2.3) yields

$$\Big[\Big[\sum_{j\in A_{i_k}} c_j\Big]^{\beta} - \Big[\sum_{j\in A_{i_{k-1}}} c_j\Big]^{\beta}\Big] \cdot A \ge \Big[\Big[\sum_{j\in D} c_j\Big]^{\beta} - \Big[\sum_{j\in D_{N\setminus\{i_k\}}} c_j\Big]^{\beta}\Big] \cdot B \tag{9.2.7}$$

where the two real numbers A, B are given by

$$A = \sum_{p=0}^{\alpha-1} \Big[\sum_{j\in A_{i_k}} c_j\Big]^{\frac{\alpha-1-p}{\alpha}} \cdot \Big[\sum_{j\in A_{i_{k-1}}} c_j\Big]^{\frac{p}{\alpha}} \text{ and}$$

$$B = \sum_{p=0}^{\alpha-1} \Big[\sum_{j\in D} c_j\Big]^{\frac{\alpha-1-p}{\alpha}} \cdot \Big[\sum_{j\in D_{N\setminus\{i_k\}}} c_j\Big]^{\frac{p}{\alpha}}$$

Note that $A \leq B$ due to the sum of increasing functions x^q, where $q > 0$. From (9.2.7), together with $A \leq B$, we conclude that (9.2.6) holds. □

Corollary 9.1. *According to the theory developed for n-person 1-concave cost games $\langle N, C\rangle$ [21], the so-called* nucleolus *cost allocation $\vec{y} = (y_i)_{i\in N} \in R^N$ for any data cost game $\langle N, C_{\mathcal{DC}}\rangle$ is given by*

$$y_i = \Delta_i(N, C_{\mathcal{DC}}) - \tfrac{1}{n} \cdot \left[\sum_{j\in N} \Delta_j(N, C_{\mathcal{DC}}) - C_{\mathcal{DC}}(N)\right] \tag{9.2.8}$$

Because $C_{\mathcal{DC}}(N) = 0$, it holds $\Delta_i(N, C_{\mathcal{DC}}) = -C_{\mathcal{DC}}(N\backslash\{i\})$ for all $i \in N$ and so, (9.2.8) simplifies as follows: for all $i \in N$,

$$y_i = -C_{\mathcal{DC}}(N\backslash\{i\}) + \frac{\Delta(N, C_{\mathcal{DC}})}{n} \quad \textit{where} \quad \Delta(N, C_{\mathcal{DC}}) = \sum_{j\in N} C_{\mathcal{DC}}(N\backslash\{j\}) \tag{9.2.9}$$

In particular, $y_i < 0$ iff $C_{\mathcal{DC}}(N\backslash\{i\}) > \frac{\Delta(N, C_{\mathcal{DC}})}{n}$. In words, according to the nucleolus, a player i receives a compensation if and only if the coalitional cost $C_{\mathcal{DC}}(N\backslash\{i\})$ strictly majorizes the average of such expressions, that is the $(n-1)$-person coalition not containing player i owns sufficiently few data.

9.3 1-Concavity of the Data Cost Game: 2nd proof

An alternative, but not less attractive proof of the main Theorem 9.1 is based on the algebraic representation of any data cost game with respect to a suitably chosen basis of the whole game space with fixed player set N. The proposed new basis has been introduced and developed in [10] as a subclass of 1-concave n-person games. In turn, we establish that every data cost game can be decomposed as a linear combination, with nonnegative coefficients, of these so-called *complementary unanimity cost games.*

Definition 9.3. [10] (with adapted notation). With every coalition $T \subseteq N$, $T \neq N$, $T \neq \emptyset$, there is associated the *complementary unanimity cost game* $\langle N, C_T\rangle$ given by

$$C_T(S) = \begin{cases} 1, & \text{if } S \neq \emptyset \text{ and } S \cap T = \emptyset; \\ 0, & \text{if } S = \emptyset \text{ or } S \cap T \neq \emptyset. \end{cases} \tag{9.3.1}$$

In addition, the complimentary unanimity cost game $\langle N, C_\emptyset \rangle$ is given by $C_\emptyset(\emptyset) = 0$ and $C_\emptyset(S) = 1$ otherwise. Note that $C_T(N) = 0$ for all $T \subsetneq N$, except $T = \emptyset$.

Remark 9.1. It is left for the reader to check that every complementary unanimity cost game $\langle N, C_T \rangle$ of the form (9.3.1), provided $T \neq \emptyset$, may be interpreted as the data cost game associated with any data sharing situation in which complementary agents own no data (i.e., $D_i = \emptyset$ for all $i \in N \backslash T$), whereas agents of T own the same data set (i.e., $D_i = D$ for all $i \in T$) with the total data cost to be normalized in that $\sum_{j \in D} c_j = 1$. As shown in [10], every complementary unanimity cost game possesses the 1-concavity property.

Theorem 9.3. *Let $\mathcal{D}C = (N, \mathcal{D}, \mathcal{C})$ be a data and cost sharing situation. For every data $j \in D$, let $N_j = \{i \in N \mid j \in D_i\}$ be the set of agents who own data j. Every data cost game $\langle N, C_{\mathcal{D}C} \rangle$ can be decomposed as the following linear combination of a number of complementary unanimity cost games with nonnegative coefficients:*

$$C_{\mathcal{D}C} = \sum_{j \in D} c_j \cdot C_{N_j} \qquad \textit{where } N_j \neq \emptyset \textit{ for all } j \in D. \tag{9.3.2}$$

Proof. Fix coalition $S \subseteq N$, $S \neq \emptyset$. We prove the equality $\sum\limits_{j \in D} c_j \cdot C_{N_j}(S) = C_{\mathcal{D}C}(S)$. For that purpose, note the following equivalences, given any data $j \in D$: (i)$C_{N_j}(S) = 1$
(ii)$S \cap N_j = \emptyset$
(iii)$i \notin N_j$, for all $i \in S$
(iv)$j \notin D_i$, for all $i \in S$
(v)$j \notin D_S$
Hence, $C_{N_j}(S) = 1$ if and only if $j \notin D_S$. Thus, $\sum\limits_{j \in D} c_j \cdot C_{N_j}(S) = \sum\limits_{j \in D \backslash D_S} c_j = C_{\mathcal{D}C}(S)$. □

Theorem 9.1 is a corollary of Theorem 9.3. In case there would exist overall missing data (that is, $D \neq D_N$), then the linear combination of the form (9.3.2) has to be extended with the additional term $c(D \backslash D_N) \cdot C_\emptyset$. The decomposition result (9.3.2) for data cost games is an extremely helpful tool to determine its Shapley cost allocation as well as its nucleolus cost allocation by exploiting the additivity property for the nucleolus on the subclass of data cost games. The cardinality of any finite set X is denoted by $|X|$.

Corollary 9.2. *As shown in [10], the well-known Shapley cost allocation charged to the agents of any n-person complementary unanimity cost game $\langle N, C_T\rangle$ amounts*

$$Sh_i(N, C_T) = \begin{cases} \frac{1}{n}, & \text{for all } i \in N\backslash T; \\ \frac{1}{n} - \frac{1}{|T|}, & \text{for all } i \in T. \end{cases}$$

By applying the additivity property of the Shapley cost allocation to the decomposition result (9.3.2), it follows that the Shapley cost allocation charged to any agent i in any data cost game $\langle N, C_{\mathcal{DC}}\rangle$ amounts $Sh_i(N, C_{\mathcal{DC}}) = \frac{c(D)}{n} - \sum_{j\in D_i} \frac{c_j}{|N_j|}$. In words, the cost $c(D)$ of the whole data set is charged equally among the agents, whereas each agent receives the individual compensation amounting the total cost of data (not necessarily uniquely) owned by the agent, in proportion with the number $|N_j|$ of owners of any data j, $j \in D$.

Corollary 9.3. *In view of (9.2.8), the nucleolus cost allocation charged to the agents of any n-person complementary unanimity cost game $\langle N, C_T\rangle$ amounts zero whenever coalition T contains at least two players (in fact, its Core degenerates into a singleton composed of the zero vector). In case T is an one-person coalition, say $T = \{i\}$, then its nucleolus cost allocation $\mu(N, C_{\{i\}})$ coincides with its Shapley cost allocation, that is $\mu_j(N, C_{\{i\}}) = \frac{1}{n}$ for all $j \in N\backslash\{i\}$, whereas $\mu_i(N, C_{\{i\}}) = \frac{1}{n} - 1$. By applying the appealing additivity property of the nucleolus cost allocation on the class of 1-concave cost games with fixed player set N, it follows from the decomposition result (9.3.2) that the nucleolus cost allocation charged to any agent i in any data cost game $\langle N, C_{\mathcal{DC}}\rangle$ amounts*

$$\mu_i(N, C_{\mathcal{DC}}) = \frac{1}{n} \cdot \sum_{\substack{j\in D,\\ |N_j|=1}} c_j - \sum_{\substack{j\in D,\\ N_j=\{i\}}} c_j \quad \text{for all } i \in N, \text{ that is}$$

$$\mu_i(N, C_{\mathcal{DC}}) = -\alpha_i + \frac{\alpha}{n} \quad \text{where} \quad \alpha_i = \sum_{\substack{j\in D,\\ N_j=\{i\}}} c_j \quad \text{and} \quad \alpha = \sum_{i\in N} \alpha_i = \sum_{\substack{j\in D,\\ |N_j|=1}} c_j$$

Notice that the latter nucleolus cost allocation is fully determined by the costs of data of unique owners. In fact, the total cost of data uniquely owned is charged equally among the agents, whereas each agent receives the individual compensation amounting the total cost of data uniquely owned by the agent.

Observe the equivalence $N_j = \{i\}$ iff $j \in D \backslash D_{N\backslash\{i\}}$ and hence,

$$\alpha_i = \sum_{\substack{j\in D,\\ N_j=\{i\}}} c_j = \sum_{j\in D\backslash D_{N\backslash\{i\}}} c_j = C_{\mathcal{DC}}(N\backslash\{i\}) \qquad \textit{in accordance with (9.2.9).}$$

Moreover, the 1*-concavity condition (9.1.2) for the data cost game $\langle N, C\rangle$ of the form (9.1.1) may be reformulated as*

$$C_{\mathcal{DC}}(S) \geq \sum_{i\in N\backslash S} \alpha_i \qquad \textit{or equivalently,} \qquad \sum_{j\in D\backslash D_S} c_j \geq \sum_{i\in N\backslash S} \sum_{\substack{j\in D,\\ N_j=\{i\}}} c_j \tag{9.3.3}$$

9.4 Concluding Remarks

Two significantly different proofs of the 1-concavity property for data cost games are treated in Sections 9.2 and 9.3. The second proof technique is based on an algebraic decomposition method which is an extremely helpful tool for the determination of the Shapley and nucleolus cost allocation. Due to 1-concavity, the formula (9.2.8) for the nucleolus cost allocation is fully determined in terms of the marginal costs $\Delta_i(N, C) = C(N) - C(N\backslash\{i\})$, $i \in N$, together with $C(N) = 0$. Result about the Core for data cost games is beyond the scope of this chapter and can be found in [7], [8], [11]. A third and fourth proof of the main Theorem 9.1 is given in [11]. One of these proofs deals with individual data sets that form a partition of the whole data set. Three other applications of one-concavity or one-convexity, called library game, co-insurance game, and the dual game of the Stackelberg oligopoly game respectively, can be found in [10], [13] and [12]. The nucleolus for 2-convex games is treated in [15]. The search for other appealing classes of cost games satisfying the 1-concavity property is still going on.

Chapter 10

Convexity of the "Airport Profit Game" and k-Convexity of the "Bankruptcy Game"

ABSTRACT - In this chapter, the topic is two-fold. Firstly, we prove the convexity of Owen's Airport Profit Game (inclusive of revenues and costs). As an adjunct, we characterize the class of 1*-convex Airport Profit Games by equivalent properties of the corresponding cost function. Secondly, we classify the class of* 1*-convex Bankruptcy Games by solving a minimization problem of its corresponding gap function.*

10.1 The Airport Profit Game: the model and its properties

In the period of Schmeidler's pioneering research on the Nucleolus [53], two previous papers [39], [38] dealt with the study of the Nucleolus for the so-called "Airport Cost Game" and "Airport Profit Game" respectively. The characteristic function of the former game was given by the negative of the airport

runway construction cost function such that the capital cost of a runway depends upon the largest aircraft type for which the runway is designed. The characteristic function of the latter game takes account of optional revenues generated by the aircraft movements such that the worth of any coalition of aircraft movements was defined as the maximum revenue less construction cost attainable by that coalition. This would allow any coalition to build a runway accommodating only a subset of its members, if that were more profitable.
In this airport runway setting, the finite set $N = \{1, 2, \ldots, n\}$ of players comprised movements (take-offs and landings) by different types of aircrafts. Define two vectors $(r_i)_{i\in N}$ and $(c_i)_{i\in N}$ where $r_i > 0$ is the revenue and $c_i > 0$ the cost associated with player i, $i \in N$. Without loss of generality, let the players be indexed in strictly increasing order such that $0 := c_0 < c_1 < c_2 < \ldots\ldots < c_{n-1} < c_n$. Define the characteristic function $v : \mathcal{P}(N) \to R$ of the *Airport Profit Game* $\langle N, v\rangle$ by $v(\emptyset) = 0$ and for all non-empty coalitions $S \subseteq N$, $S \neq \emptyset$,

$$v(S) \quad = \quad \max_{k\in\{1,2,\ldots,n\}} \left[\sum_{\substack{j\in S,\\ 1\leq j\leq k}} r_j - c_k \right] \quad \text{or equivalently,} \tag{10.1.1}$$

$$v(S) \quad = \quad \max_{k\in\{1,2,\ldots,s\}} \left[\sum_{j=1}^{k} r_{i_j} - c_{i_k} \right] \tag{10.1.2}$$

Throughout the remainder of this chapter, re-number the members of any non-empty coalition S according to the order of players such that
$S = \{i_1, i_2, \ldots, i_{s-1}, i_s\}$, where $s = |S|$ denotes the cardinality of the finite set S. Here player i_1 is the smallest player in S, i_2 is the smallest but one, and i_s is the largest player in S, also denoted by i_S. In case $1 \notin S$, then the maximization problem (10.1.2) is meant to start with $-c_1$ and otherwise, to start with $r_1 - c_1$. Generally speaking, the worths of one-person coalitions are of two types, namely $v(\{1\}) = r_1 - c_1$ as well as $v(\{i\}) = \max\left[-c_1, \quad r_i - c_i\right]$ for all $i \in N\backslash\{1\}$. It is left to the reader to check the validity of the superadditivity property in that $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. By (10.1.3), the individual benefit of any player is bounded above by the player's revenue and bounded below by zero.

Proposition 1. *The Airport Profit Game $\langle N, v\rangle$ satisfies the following properties.*

$$(i) \quad 0 \le v(S \cup \{i\}) - v(S) \le r_i \quad \text{for all } S \subseteq N, \text{ all } i \in N \backslash S. \tag{10.1.3}$$

$$(ii) \quad v(S) = \max\left[v(S\backslash\{i_S\}), \quad \sum_{j \in S} r_j - c_{i_S}\right] \quad \text{if } s \ge 2 \tag{10.1.4}$$

$$(iii) \quad v(T) = v(T\backslash\{1\}) + r_1 \quad \text{for all } T \subseteq N,\ 1 \in T,\ T \neq \{1\} \tag{10.1.5}$$

$$(iv) \quad r_i \ge v(\{i\}) + c_1 \quad \text{for all } i \in N,\ i \neq 1 \tag{10.1.6}$$

$$(v) \quad v(\{i,j\}) \ge v(\{i\}) + v(\{j\}) + c_1 \quad \text{for all } i, j \in N,\ i \neq j \tag{10.1.7}$$

Proof. (i) Let $S \subseteq N\backslash\{1\}$, and $i \in N\backslash S$. Then the boundedness of the player's individual benefit $v(S \cup \{i\}) - v(S)$ from below by zero and from above by r_i respectively, is a direct consequence of, on the one hand, the maximization problem to attain $v(S)$ as the maximum term in some sequence α_k, $k \in S$, and, on the other hand, the maximization problem to attain $v(S \cup \{i\})$ as the maximum term in the corresponding extended sequence $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta, \alpha_{p+1} + r_i, \alpha_{p+2} + r_i, \ldots, \alpha_s + r_i$. Here the intermediate term β amounts $\alpha_p + c_p + r_i - c_i$. Independently of β, the lower inequality $v(S \cup \{i\}) \ge v(S)$ holds, while the upper inequality $v(S \cup \{i\}) - v(S) \le r_i$ holds because the corresponding entries in both sequences do differ at most the amount r_i, in particular the entry β in the second sequence differs from the entry α_p in the first sequence in that the amount $\beta - \alpha_p = c_p - c_i + r_i < r_i$ since $c_i > c_p$ by choosing p as the direct predecessor of i in S.
In case player i is the largest player of the coalition $S \cup \{i\}$, then the tail of both sequences with corresponding differences amounting r_i vanish and the identical heading of both sequences yields the significant relationship $v(S \cup \{i\}) = \max\left[v(S), \sum_{j \in S\cup\{i\}} r_j - c_i\right]$, provided $i = i_{S\cup\{i\}}$. The study of the remaining case in which $i < k$ for all $k \in S$ is left to the reader. This proves (10.1.3) as well as (10.1.4). Further, (10.1.5) is a direct consequence of (10.1.2). Moreover, we prove (10.1.6) in case $i \neq 1$ through the following

chain of equivalences.

$$r_i \geq v(\{i\}) + c_1 \iff r_i \geq c_1 + \max\Big[-c_1, \quad r_i - c_i\Big]$$
$$\iff r_i \geq \max\Big[0, \quad r_i - c_i + c_1\Big]$$
$$\iff r_i \geq r_i - c_i + c_1 \iff c_i \geq c_1$$

The latter inequality $c_i \geq c_1$ holds by ordering the costs. So, (10.1.6) holds. Finally, we prove (10.1.7) in case $i < j$ through the following chain of equivalences.

$$v(\{i,j\}) \geq v(\{i\}) + v(\{j\}) + c_1$$
$$\iff v(\{i,j\}) - v(\{i\}) \geq v(\{j\}) + c_1 \qquad \text{assuming } i < j$$
$$\iff \max\Big[0, \quad r_i + r_j - c_j - v(\{i\})\Big] \geq \max\Big[0, \quad r_j - c_j + c_1\Big]$$
$$\Longleftarrow r_i + r_j - c_j - v(\{i\}) \geq r_j - c_j + c_1$$
$$\iff r_i \geq v(\{i\}) + c_1$$

So, (10.1.7) holds, provided (10.1.6) holds. □

For instance, the worths of multi-person coalitions in any three-person Airport Profit Game $\langle N, v\rangle$ are given by

$$v(\{1,2\}) = \max\Big[r_1 - c_1, \quad r_1 + r_2 - c_2\Big]$$
$$v(\{1,3\}) = \max\Big[r_1 - c_1, \quad r_1 + r_3 - c_3\Big]$$
$$v(\{2,3\}) = \max\Big[-c_1, \quad r_2 - c_2, \quad r_2 + r_3 - c_3\Big] \qquad \text{as well as}$$
$$v(\{1,2,3\}) = \max\Big[r_1 - c_1, \quad r_1 + r_2 - c_2, \quad r_1 + r_2 + r_3 - c_3\Big]$$

Observe that player 1 acts as a quasi-dummy player in that each individual benefit $v(T) - v(T\backslash\{1\}) = r_1$ for all coalitions T containing player 1, except for the singleton $T = \{1\}$. In fact, this significant relationship (10.1.5) yields in a straightforward manner the quasi-dummy player property for player 1 in any n-person Airport Profit Game and consequently, the payoff according to the Shapley value of player 1 equals $r_1 - \frac{c_1}{n}$. An alternative proof of (10.1.5) proceeds by induction on the coalition size and is omitted.

10.2 The convexity of the Airport Profit Game

The most significant property of the Airport Profit Game is its convexity. In this section, we will study the convexity property of the airport profit game. Before that we recall the definitions of convex game.

Generally speaking, a cooperative game $\langle N, v\rangle$ with arbitrary characteristic function $v : \mathcal{P}(N) \to R$ is said to be *convex* if it holds [54]

$$\begin{array}{ll}
(i) & v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \text{for all } S, T \subseteq N \\
(ii) & v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \\
& \text{for all } S, T \subseteq N,\ S \subseteq T,\ i \in N \backslash T, \text{ or} \\
(iii) & v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}) \\
& \text{for all } S \subseteq N,\ i, j \in N \backslash S, \text{ or} \\
(iv) & \Delta_i^v(S) \leq \Delta_i^v(S \cup \{j\}) \quad \text{for all } S \subseteq N,\ i, j \in N \backslash S,\ i \neq j, \qquad (10.2.1) \\
& \text{where} \Delta_k^v(T) = v(T \cup \{k\}) - v(T) \quad \text{for all } T \subseteq N, \text{ and all } k \in N \backslash T.
\end{array}$$

Theorem 10.1. *(i) If $1 < i < j \leq n$ and $\Delta_i^v(T \cup \{j\}) \neq r_i$, then $\Delta_j^v(T) = 0$*
(ii) The Airport Profit Game $\langle N, v\rangle$ satisfies the convexity property (10.2.1)

Proof. (i) Let $1 < i < j \leq n$ and $T \subseteq N$ such that $\Delta_i^v(T \cup \{j\}) \neq r_i$. In words, this means that there is no maximizer to attain the worth $v(T \cup \{j\})$ in the tail of the corresponding maximization problem and what is left, its simultaneous maximizer is attained in the heading of the two maximization problems for $T \cup \{j\}$ and T respectively, stated as $\Delta_j^v(T) = 0$.

Formally, we distinguish two cases. If $i_1 < i$, then there exists $1 \leq k \leq p$ such that $v(T \cup \{j\}) = \alpha_k$. By feasibility, $\alpha_k \leq v(T)$ and in turn, by monotonicity

(10.1.3), it holds $v(T) = v(T \cup \{j\})$.
If $i_1 > i$, then $v(T \cup \{j\}) = -c_1$ and once again by feasibility and monotonicity, it holds $v(T \cup \{j\}) = v(T)$. In both cases, it holds $\Delta_j^v(T) = 0$. This proves the relevant implication.
(ii) Firstly, (10.2.1) holds by super-additivity when $S = \emptyset$. Assume $S \neq \emptyset$. Secondly, if $i = 1$ (or $j = 1$), then, by (10.1.5), the equalities will be met in (10.2.1). Assume $1 \notin \{i, j\}$. Thirdly, in case $S = \{1\}$, check (10.2.1) through the following equivalence and implication (once again using (10.1.5)):

$$\begin{aligned} & v(\{1,i,j\}) - v(\{1,i\}) \geq v(\{1,j\}) - v(\{1\}) \\ \iff \quad & v(\{i,j\}) - v(\{i\}) \geq v(\{j\}) + c_1 \qquad \text{assuming } i < j \\ \Longleftarrow \quad & r_i \geq v(\{i\}) + c_1 \end{aligned}$$

By Proposition1(iv-v), the convexity condition (10.2.1) holds. Fourthly, in case $1 \in S$, $S \neq \{1\}$, then player 1 is a member of all four coalitions, each of which with coalition size at least two, and so, (10.2.1) reduces to the following condition:

$$v(T \cup \{i\}) - v(T) \leq v(T \cup \{i,j\}) - v(T \cup \{j\}) \quad \text{where } T = S \backslash \{1\} \quad (10.2.2)$$

So far, it remains to check (10.2.2) only for coalitions T not containing player 1. Fifthly, we distinguish two cases. If $v(T \cup \{j\}) = v(T)$, then (10.2.2) holds by the monotonicity of the game $\langle N, v \rangle$ derived from (10.1.3) and thus, $v(T \cup \{i,j\}) \geq v(T \cup \{i\})$. If $v(T \cup \{j\}) > v(T)$, i.e., $\Delta_j^v(T) > 0$, then part (i) implies $\Delta_i^v(T \cup \{j\}) = r_i$. Together with (10.1.3) once again, we derive the chain of (in)equalities:

$$v(T \cup \{i,j\}) - v(T \cup \{j\}) = \Delta_i^v(T \cup \{j\}) = r_i \geq v(T \cup \{i\}) - v(T)$$

This completes the full proof of convexity (10.2.2) of the Airport Profit game $\langle N, v \rangle$. □

Remark 10.1. In the framework of the convexity constraint $v(\{3,4,5\}) - v(\{3,5\}) \geq v(\{4,5\}) - v(\{5\})$ of the form (10.2.2) applied with $i = 4$, $j = 3$, $T = \{5\}$, it violates our assumption $i < j$. Due to the symmetrical roles of both players i and j in (10.2.2), we interchange both worths in that $v(\{3,4,5\}) - v(\{4,5\}) \geq v(\{3,5\}) - v(\{5\})$, of the form (10.2.2) applied with

$i = 3$, $j = 4$, $T = \{5\}$, fulfilling the assumption $i < j$. According to the above proof technique, the latter convexity constraint is solved in two steps. In case of the assumption $v(\{4,5\}) - v(\{5\}) = 0$, the latter constraint is solved by monotonicity. Otherwise, the following implication applies: $\Big[\Delta_j^v(T) > 0\Big] \Longrightarrow \Big[\Delta_i^v(T \cup \{j\}) = r_i\Big]$, that is $\Big[\Delta_4^v(\{5\}) > 0\Big] \Longrightarrow \Big[\Delta_3^v(\{4,5\}) = r_3\Big]$, that is $v(\{4,5\}) - v(\{5\}) > 0$ implies $v(\{3,4,5\}) - v(\{4,5\}) = r_3$. □

10.3 Characterizations of 1-convexity for Airport Profit Games

Following Shapley's pioneering research on convex games [54], the adapted notion of 1-convexity requires that the characteristic function $v : \mathcal{P}(N) \to R$ satisfies a system of inequalities such that, for any non-trivial coalition S, its coalitional contribution to the formation of the grand coalition N majorizes the sum of all the individual benefits to the formation of the grand coalition by the members of the coalition S, i.e., it holds [21]

$$v(N) - v(N \backslash S) \geq \sum_{j \in S} \Big[v(N) - v(N \backslash \{j\})\Big] \quad \text{for all } S \subseteq N,\ S \neq \emptyset,\ S \neq N, \text{ i.e.,} \tag{10.3.1}$$

$$0 \leq g^v(N) \leq g^v(S) \quad \text{for all } S \subseteq N,\ S \neq \emptyset \tag{10.3.2}$$

An arbitrary cooperative game $\langle N, v \rangle$ is said to be 1-convex if the non-negative gap function attains its minimum at the grand coalition N. Generally speaking, n-person and $(n-1)$-person coalitions have the same gap, i.e., $g^v(N \backslash \{i\}) = g^v(N)$ for all $i \in N$.

The goal of this Section is to present a list of necessary and sufficient conditions on the revenues and costs for the n-person Airport Profit Game to be 1-convex. Recall $b_i^v = v(N) - v(N \backslash \{i\})$ for all $i \in N$. An essential property of the gap function for Airport Profit Games is the redundancy of the presence of player 1. That is, $g^v(S) = g^v(S \backslash \{1\})$ for all $S \subseteq N$ with $1 \in S$, $S \neq \{1\}$, because $b_1^v = r_1 = v(S) - v(S \backslash \{1\})$ by applying (10.1.5) twice. Moreover, $g^v(\{1\}) = b_1^v - v(\{1\}) = b_1^v - r_1 + c_1 = c_1$. By the latter two properties, for

three- and four-person games it holds
(i) A three-person Airport Profit Game is 1-convex if and only if $0 \leq g^v(N) \leq c_1$ (since $g^v(\{2\}) = g^v(\{1,2\}) = g^v(N)$ as well as $g^v(\{3\}) = g^v(\{1,3\}) = g^v(N)$)
(ii) A four-person Airport Profit Game is 1-convex if and only if $0 \leq g^v(N) \leq g^v(\{i\})$ for all $i \in N$ (since $g^v(\{2,3\}) = g^v(\{1,2,3\}) = g^v(N)$ etc. and $g^v(\{1,2\}) = g^v(\{2\})$ etc.)

Theorem 10.2. *The following statements for the Airport Profit Game $\langle N, v\rangle$ are equivalent.*
(i) The game is 1-convex, i.e., $0 \leq g^v(N) \leq g^v(S)$ for all $S \subseteq N$, $S \neq \emptyset$
(ii) The gap function $g^v : \mathcal{P}(N) \to R$ is constant, i.e., $g^v(S) = g^v(N)$ for all $S \subseteq N$, $S \neq \emptyset$
(iii) $g^v(S) = c_1$ for all $S \subseteq N$, $S \neq \emptyset$
(iv) $g^v(S) = c_1$ for all $S \subseteq N\backslash\{1\}$, $S \neq \emptyset$
(v) $g^v(N) = c_1$
(vi) $0 \leq g^v(N) \leq c_1$
(vii) $v(S) = \sum_{j\in S} b_j^v - v(S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Proof. Recall $b_1^v = v(N) - v(N\backslash\{1\}) = r_1$ by (10.1.5). Thus, $g^v(\{1\}) = b_1^v - v(\{1\}) = c_1$.
Part (vii) states that the Airport Profit Game is 1-convex if and only if it is strategically equivalent with the constant game leveled at $-c_1$ (where the additive part is caused by the vector of individual benefits). Notice that the convexity of the Airport Profit Game implies the monotonicity of its corresponding gap function since the particular convexity constraint $v(S \cup \{i\}) - v(S) \leq v(N) - v(N\backslash\{i\}) = b_i^v$ is equivalent to $g^v(S \cup \{i\}) \geq g^v(S)$ for all $i \in N$ and all $S \subseteq N\backslash\{i\}$. Given all these facts, it is very simple to check the equivalence of all seven statements, thus, we omit the proof here. □

Example 10.1. Consider the n-person Airport Profit Game with unitary revenues $r_i = 1$ for all $i \in N\backslash\{1\}$, except for the arbitrary revenue $r_1 \in R$, and the linearly increasing costs $c_s = s$ for all $1 \leq s \leq n$. We claim that the characteristic function $v : \mathcal{P}(N) \to R$ of this Airport Profit Game is given by $v(S) = r_1 - 1$ if $1 \in S$ and $v(S) = -1$ otherwise. We check the worth $v(S)$ for any coalition $S \subseteq N$. Concerning the singletons, $v(\{1\}) = r_1 - c_1 = r_1 - 1$, whereas $v(\{i\}) = \max[-c_1, \quad r_i - c_i] = \max[-1, \quad 1 - i] = -1$ for all $i \neq 1$. We proceed by induction on the coalition size. In case $1 \in S$ with $s \geq 2$, then

$1 \in S\backslash\{i_S\}$, and so, by induction hypothesis, $v(S\backslash\{i_S\}) = r_1 - 1$ and in turn, by (10.1.4),

$$v(S) = \max\left[v(S\backslash\{i_S\}), \quad \vec{r}(S) - c_{i_S}\right] = \max\left[r_1 - 1, \quad r_1 + (s-1) - c_{i_S}\right] = r_1 - 1$$

since $i_S \geq s$ whenever $1 \in S$. In case $1 \notin S$, then also $1 \notin S\backslash\{i_S\}$, and so, by induction hypothesis, $v(S\backslash\{i_S\}) = -1$ and in turn,
$v(S) = \max\left[v(S\backslash\{i_S\}), \quad \vec{r}(S) - c_{i_S}\right] = \max[-1, \quad s - c_{i_S}] = -1$ since $i_S \geq s + 1$ whenever $1 \notin S$. Finally, the individual benefit of any player i equals $b_i^v = v(N) - v(N\backslash\{i\}) = 0$ for all $i \in N\backslash\{1\}$, where $b_1^v = r_1$, and in turn, its gap function g^v is constant at level one, i.e., $g^v(S) = 1$ for all $S \subseteq N$, $S \neq \emptyset$. Particularly, this Airport Profit Game is 1-convex.

10.4 Bankruptcy Problem and Bankruptcy Game: Notions

A *bankruptcy problem* arises when a person dies, leaving *debts* $d_1, d_2, \ldots, d_n$, totalling at least as much as the *estate* E, i.e., $0 \leq E \leq \sum_{j=1}^{n} d_j$. The problem is that the debts are mutually inconsistent in that the estate E is insufficient to meet all of the debts. Without loss of generality, the creditors may be ordered such that $0 = d_0 \leq d_1 \leq d_2 \leq \ldots \leq d_{n-1} \leq d_n$ (otherwise renumber the heirs). Formally, a bankruptcy problem is defined as an ordered pair $\langle E, \vec{d}\rangle$ with estate E as well as the vector $\vec{d} = (d_j)_{j=1}^{n} \in R^n$ of which the coordinates are given by the debts [48]. Throughout their paper, the estate is treated like a *variable*, varying from zero till the sum of the debts.

Definition 10.1. [1] With every bankruptcy problem $\langle E, \vec{d}\rangle$, there is associated the n-person bankruptcy game $\langle N, v_{E,\vec{d}}\rangle$ of which the finite set N consists of the n heirs (creditors) and the *worth* $v_{E,\vec{d}}(S)$ of *coalition* $S \subseteq N$ equals either zero or what is left of the estate after each member $j \in N\backslash S$ of the complementary coalition $N\backslash S$ receives the debt d_j, that is

$$v_{E,\vec{d}}(S) = \max\left[0, \quad E - \sum_{j \in N\backslash S} d_j\right] \qquad \text{for all } S \subseteq N, \text{ where } \quad v_{E,\vec{d}}(N) = E \tag{10.4.1}$$

We prefer to introduce, opposite to the estate, the variable *surplus* $\epsilon = \sum_{j=1}^{n} d_j - E \geq 0$ over all the debts. So, the surplus ϵ continuously increases from the bottom zero-level up to the top level $\sum_{j=1}^{n} d_j$ as the estate decreases from the top level down to the bottom zero-level. In this alternative setting, the characteristic function of the bankruptcy game $\langle N, v_{E,\vec{d}} \rangle$ may be rewritten as

$$v_{E,\vec{d}}(S) = \max\left[0, \quad \sum_{j \in S} d_j - \epsilon\right] \qquad \text{for all } S \subseteq N \tag{10.4.2}$$

Without loss of generality, it is tacitly assumed that the smallest debt differs from zero, i.e., $d_1 > 0$. Indeed, if $d_1 = 0$, then player 1 is a so-called *null-player* in the bankruptcy game $\langle N, v_{E,\vec{d}} \rangle$ satisfying $v_{E,\vec{d}}(S \cup \{1\}) = v_{E,\vec{d}}(S)$ for all $S \subseteq N \backslash \{1\}$), and null-players are supposed to receive no award. Clearly, the bankruptcy game satisfies monotonicity and moreover, it is well-known that the bankruptcy game satisfies the convexity property 10.2.1. The main goal of this Section is to investigate slightly adapted notions called k-convexity [21], with clear affinities to convexity. We search for necessary and sufficient conditions on the estate and the debts for the bankruptcy game to be k-convex.

Definition 10.2. Let $k \in \{1, 2, \ldots, n\}$. The n-person game $\langle N, v \rangle$ is said to be *k-convex* if its corresponding gap function $g^v : \mathcal{P}(N) \to R$ satisfies the following three conditions [21]:

$$g^v(S) \geq g^v(N) \geq g^v(T) \qquad \text{for all } S, T \subseteq N \text{ with } |S| \geq k,\ |T| = k - 1 \tag{10.4.3}$$

$$g^v(T \cup \{i\}) - g^v(T) \leq g^v(S \cup \{i\}) - g^v(S), \tag{10.4.4}$$

for all $i \in N$, $S \subseteq T \subseteq N \backslash \{i\}$, $|T| \leq k - 2$

$$g^v(N) - g^v(T) \leq g^v(S \cup \{i\}) - g^v(S) \quad \text{for all } i \in N,\ S \subseteq T \subseteq N \backslash \{i\},\ |T| = k - 1 \tag{10.4.5}$$

In the setting of the sequential formation of the grand coalition N, the k-convexity conditions (10.4.4)-(10.4.5) for the game v resemble the *concavity* condition for the corresponding gap function g^v, on the understanding that individuals join the sequential formation one by one till coalition size $k - 1$, whereas the last $n + 1 - k$ players merge as one syndicate. By (10.4.3), gaps of coalitions with at least k players weakly majorize the gap of the grand coalition N, while on its turn, the gap of the grand coalition weakly majorizes the gaps of coalitions of size $k-1$. Notice that (10.4.3)-(10.4.4), applied to $k=1$, agree with $1-$convexity condition. Moreover, k-convexity for n-person games, applied to $k=n$, fully agrees with the convexity condition. Generally speaking, k-convexity and m-convexity are not compatible whenever $k \neq m$ because of different levels for the gap of the grand coalition. For the bankruptcy game, however, due to its convexity, we shall show that several notions of k-convexity are compatible.

10.5 k-Convexity of Bankruptcy Games: approach by figures

The forthcoming main result states necessary and sufficient conditions for the k-convexity of the (convex) bankruptcy game. The proof technique is based on depicting the table of the gap $g^v(S)$ of any non-empty coalition S as a function of the variable surplus ϵ on its domain $[0, \vec{d}(N)]$. In fact, we distinguish four types of coalitions, including the grand coalition. Throughout the sequel, write v instead of $v_{E,\vec{d}}$ and for any coalition S, put $\vec{d}(S) = \sum_{j\in S} d_j$, $\vec{b}^v(S) = \sum_{j\in S} b_j^v$ as well as $g^v(S) = \vec{b}^v(S) - v(S)$, where $b_i^v = v(N) - v(N\backslash\{i\})$ for all $i \in N$. Put $\vec{d}(\emptyset) = 0 = \vec{b}^v(\emptyset)$. Note that the convexity of a game $\langle N, v\rangle$ implies the monotonicity of its gap function g^v in that $g^v(S\cup\{i\}) \geq g^v(S)$ for all $i \in N$, all $S \subseteq N\backslash\{i\}$ (due to the convexity condition $v(S\cup\{i\}) - v(S) \leq v(N) - v(N\backslash\{i\})$).

Theorem 10.3. *Consider the n-person bankruptcy problem with estate $E > 0$ and ordered debts $0 = d_0 < d_1 \leq d_2 \leq \ldots \leq d_{n-1} \leq d_n$. Put $\epsilon = \sum_{j=1}^n d_j - E$ and let $1 \leq k \leq n-2$. Then the corresponding bankruptcy game $\langle N, v_{E,\vec{d}}\rangle$ of the form (10.4.2) is k-convex if and only if*

$$0 \leq \epsilon \leq \sum_{j=1}^{k} d_j \qquad \text{or equivalently,} \qquad \sum_{j=k+1}^{n} d_j \leq E \leq \sum_{j=1}^{n} d_j \qquad (10.5.1)$$

The tables 10.1,10.2 and 10.3 show the relationship among surplus ϵ, individual benefit $b_i^v, i \in N$ and gap function which measures the surplus of the individual benefit of its members over its worth. The calculations are simple,

Table 10.1: The gap $g^v(S)$ as a function of ϵ.

surplus ϵ	b_i^v	**gap** $g^v(S)$	**gap** $g^v(N)$
$\epsilon \leq \vec{d}(N\backslash\{n\})$	d_i	$\min[\vec{d}(S), \epsilon]$	ϵ
$\epsilon > \vec{d}(N\backslash\{n\})$	$\min\left[\vec{d}(N) - \epsilon, d_i\right]$	$----$	$\vec{d}(N\backslash\{n\}) + \sum_{j\in N, j\neq n} \min\left[\vec{d}(N\backslash\{j\}) - \epsilon, 0\right]$

Table 10.2: Consider coalition S satisfying $\vec{d}(S) > \vec{d}(N\backslash\{n\})$.

surplus ϵ	b_i^v	**gap** $g^v(S)$
$\vec{d}(N\backslash\{n\}) < \epsilon \leq \min\left[\vec{d}(S), \vec{d}(N\backslash\{n-1\})\right]$	$d_i, i \in S, i \neq n$	$\vec{d}(N\backslash\{n\})$

Table 10.3: Consider the k-person coalition $S_k = \{1, 2, \ldots, k\}$ where $1 \leq k \leq n-2$.

surplus ϵ	b_i^v	**gap** $g^v(S_k)$
$\vec{d}(N\backslash\{n-1, n\}) < \epsilon \leq \vec{d}(N\backslash\{n\})$	d_i	$g^v(S_k) = \vec{d}(S_k) < \epsilon = g^v(N)$
$\vec{d}(N\backslash\{n\}) \leq \epsilon < \vec{d}(N)$	$\min\left[\vec{d}(N) - \epsilon, d_i\right], i \neq n$	$g^v(S_k) < g^v(N)$

thus, we omit them here.

Proposition 2. *Firstly we describe the figure of the gap of the grand coalition as a function of the variable surplus* ϵ*. For that purpose, partition the domain* $[0, \vec{d}(N)]$ *into the intervals* I_m*,* $0 \leq m \leq n$*, where the* m*-th interval is given by* $I_m = (\vec{d}(N\backslash\{n+1-m\}), \vec{d}(N\backslash\{n-m\})]$*. Put* $\vec{d}(N\backslash\{n+1\}) = 0$ *as well as* $\vec{d}(N\backslash\{0\}) = \vec{d}(N)$ *with reference to the left- and right- endpoint of the domain. Concerning the grand coalition* N*, it follows from the second line of Table 10.1 that the gap* $g^v(N)$ *is a linearly increasing function of* ϵ*, with slope* 1*, on the first interval* I_0*.*
By the second line of Table 10.1, $g^v(N)$ *is constant at the level* $\vec{d}(N\backslash\{n\})$ *on the second interval* I_1*.*
By the third line of Table 10.1, $g^v(N)$ *is a piecewise linearly decreasing function on any interval* I_m*,* $2 \leq m \leq n$*, with slope the negative sign of* $m-1$*. For instance,* $g^v(N) = (n-1)\cdot d_1$ *if* $\epsilon = \vec{d}(N\backslash\{1\})$*, whereas* $g^v(N) = (n-1)\cdot d_1 + (n-2)\cdot(d_2 - d_1) = d_1 + (n-2)\cdot d_2$ *if* $\epsilon = \vec{d}(N\backslash\{2\})$*.*

Proposition 3. *Let* $S \subseteq N$*,* $S \neq \emptyset$*. For notation' convenience, without loss of generality, put* $S = \{i_1, i_2, \ldots, i_s\}$ *such that* $d_{i_1} \leq d_{i_2} \leq \ldots \leq d_{i_s}$*. Put* $i_0 = 0$

and $d_0 = 0$.

Case one. *Concerning any coalition* S *satisfying* $\vec{d}(S) \leq \vec{d}(N\backslash\{n\})$, *partition the domain into the intervals* J_m, $0 \leq m \leq s+1$, *where the first interval* $J_0 = (0, \vec{d}(S)]$, *the second interval* $J_1 = (\vec{d}(S), d(N\backslash\{i_s\})]$, *and the* m*-th interval,* $2 \leq m \leq s+1$, *is given by* $J_m = (\vec{d}(N\backslash\{i_{s+2-m}\}), \vec{d}(N\backslash\{i_{s+1-m}\})]$. *Put* $\vec{d}(N\backslash\{i_0\}) = \vec{d}(N)$.

On the first interval J_0, *the gap* $g^v(S)$ *is a linearly increasing function of* ϵ, *with slope* 1, *since* $g^v(S) = \vec{b}^v(S) - v(S) = \vec{b}^v(S) - \vec{d}(S) + \epsilon = \epsilon$.

The gap $g^v(S)$ *is constant at the level* $\vec{d}(S)$ *on the second interval* J_1.

The gap $g^v(S)$ *is a piecewise linearly decreasing function on any interval* J_m, $2 \leq m \leq s+1$, *with slope the negative sign of* $m-1$. *For instance,* $g^v(S) = s \cdot d_{i_1}$ *if* $\epsilon = \vec{d}(N\backslash\{i_1\})$, *whereas* $g^v(S) = s \cdot d_{i_1} + (s-1) \cdot (d_{i_2} - d_{i_1}) = d_{i_1} + (s-1) \cdot d_{i_2}$ *if* $\epsilon = \vec{d}(N\backslash\{i_2\})$.

The constant level $\vec{d}(S)$ *on the second interval* J_1 *is caused on the one hand by* $v(S) = 0$ *and on the other by the coincidence* $b_i^v = d_i$ *for all* $i \in S$ *due to Table 10.1.*

Case two. *Concerning any coalition* S *satisfying* $\vec{d}(S) > \vec{d}(N\backslash\{n\})$, *note that* $n \in S$, *write* $i_s = n$, *and, due to the partition* J_m, $2 \leq m \leq s+1$, *of the partial domain* $(\vec{d}(N\backslash\{n\}), \vec{d}(N)]$, *there exists a unique* $1 \leq t \leq s$ *such that* $\vec{d}(S) \in J_{s+2-t}$.

On the first interval $[0, \vec{d}(N\backslash\{n\})]$, *the gap* $g^v(S)$ *is a linearly increasing function of* ϵ, *with slope* 1, *because of the coincidence* $\vec{b}^v = \vec{d}$ *and* $v(S) = \vec{d}(S) - \epsilon$. *By Table 10.2, the gap* $g^v(S)$ *continues to be constant for a while at level* $\vec{d}(N\backslash\{n\})$ *and next piecewise linearly decreasing till level zero. For its exact description, we distinguish two subcases.*

Subcase one. *Suppose* $\vec{d}(S) \leq \vec{d}(N\backslash\{n-1\})$. *On the first sub-interval* $[\vec{d}(N\backslash\{n\}), \vec{d}(S)]$ *of interval* J_2, *the gap* $g^v(S)$ *attains its maximum at constant level* $\vec{d}(N\backslash\{n\})$, *and is linearly decreasing with slope* 1 *on the second sub-interval* $[\vec{d}(S), \vec{d}(N\backslash\{i_{s-1}\})]$ *of interval* J_2 *(including the intermediate* $\epsilon = \vec{d}(N\backslash\{n-1\})$*). On the remaining intervals* J_m, $3 \leq m \leq s+1$, *the gap* $g^v(S)$ *is still linearly decreasing with slope* $m-1$, *varying from slope* 2 *on* J_3 *up to slope* s *on* J_{s+1}.

Subcase two. *Suppose* $\vec{d}(S) > \vec{d}(N\backslash\{n-1\})$. *Notice that* $n-1 \in S$. *On the full interval* J_2, *the gap* $g^v(S)$ *attains its maximum at constant level* $\vec{d}(N\backslash\{n\})$. *On the intervals* J_m, $3 \leq m \leq s+1-t$, *the gap* $g^v(S)$ *is piecewise linearly decreasing with slope* $m-2$, *varying from slope* 1 *on* J_3 *up to slope* $s-1-t$ *on* J_{s+1-t}. *Interval* J_{s+2-t} *is divided into two sub-intervals left and right from its intermediate* $\epsilon = \vec{d}(S)$ *with slopes* $s-t$ *and* $s+1-t$ *respectively. On the remaining intervals* J_m, $s+3-t \leq m \leq s+1$, *the gap* $g^v(S)$ *is still piecewise*

Table 10.4: The gaps $g^v(S_k)$ and $g^v(N)$ as a piecewise function of ϵ.

surplus ϵ	**gap** $g^v(S_k)$	**gap** $g^v(N)$
$\vec{d}(N)$	0	0
$\vec{d}(N\backslash\{1\})$	$k \cdot d_1$	$(n-1) \cdot d_1$
$\vec{d}(N\backslash\{2\})$	$k \cdot d_1 + (k-1) \cdot (d_2 - d_1)$	$(n-1) \cdot d_1 + (n-2) \cdot (d_2 - d_1)$
	$= d_1 + (k-1) \cdot d_2$	$= d_1 + (n-2) \cdot d_2$
$\vec{d}(N\backslash\{3\})$	$d_1 + (k-1) \cdot d_2 + (k-2) \cdot (d_3 - d_2)$	$d_1 + (n-2) \cdot d_2 + (n-3) \cdot (d_3 - d_2)$
	$= d_1 + d_2 + (k-2) \cdot d_3$	$= d_1 + d_2 + (n-3) \cdot d_3$
$\vec{d}(N\backslash\{k-1\})$	$d_1 + d_2 + \ldots + d_{k-2} + 2 \cdot d_{k-1}$	$d_1 + \ldots + d_{k-2} + (n+1-k) \cdot d_{k-1}$
$\vec{d}(N\backslash\{k\})$	$\vec{d}(S_k)$	$\vec{d}(S_k) + (n-1-k) \cdot d_k$
$\vec{d}(N\backslash\{n-1\})$	$\vec{d}(S_k)$	$\vec{d}(N\backslash\{n\})$
$\vec{d}(N\backslash\{n\})$	$\vec{d}(S_k)$	$\vec{d}(N\backslash\{n\})$
$\vec{d}(S_k)$	$\vec{d}(S_k)$	$\vec{d}(S_k)$

linearly decreasing with slope $m-1$, varying from slope $s+2-t$ on J_{s+3-t} up to slope s on J_{s+1}.
Concerning the grand coalition $S = N$, the latter second subcase applies with $s = n$, $t = 1$, and reduces to the former Proposition 2

10.6 Proof of Theorem 10.3

We distinguish a necessity and sufficiency part. Put $S_k = \{1, 2, \ldots, k\}$ where $1 \leq k \leq n-2$.

Proof of necessity part. From a detailed comparison of the gaps $g^v(S_k)$ and $g^v(N)$, depicted as functions of the variable surplus ϵ, the figures show that the tail of $g^v(N)$ strictly majorizes the tail of $g^v(S_k)$ (except for the degenerated case $\epsilon = \vec{d}(N)$ yielding the zero game). For instance, on the last interval I_n, the slope $n-1$ of $g^v(N)$ strictly majorizes the slope k of $g^v(S_k)$ and similarly, on the last but one interval I_{n-1}, the slope $n-2$ strictly majorizes the slope $k-1$ resulting in the levels $d_1 + (n-2) \cdot d_2$ versus $d_1 + (k-1) \cdot d_2$, and so on. The reader is invited to study the next Table.

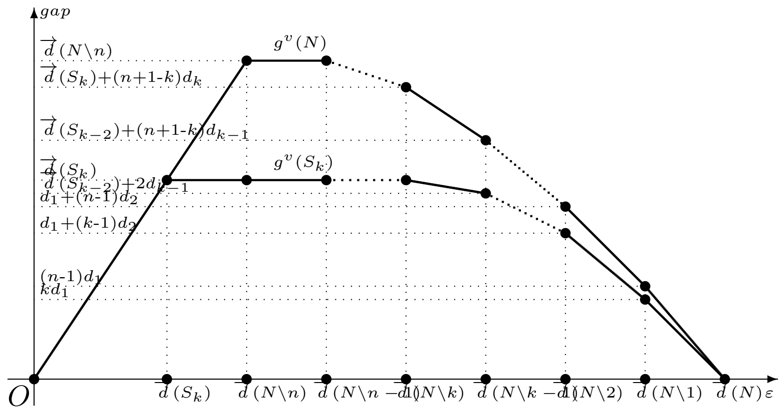


Figure 1: The gaps $g^v(N)$ and $g^v(S_k)$ as a piecewise function of the variable surplus ϵ

Through Figure 1 and Table 10.4, we conclude that $g^v(N) > g^v(S_k)$ whenever $\epsilon > \vec{d}(S_k)$, $\epsilon \neq \vec{d}(N)$. Thus, the k-convexity condition $g^v(S_k) \geq g^v(N)$ for the k-person coalition S_k yields the necessary condition $0 \leq \epsilon \leq \vec{d}(S_k)$ (ignoring the degenerated case $\epsilon = \vec{d}(N)$). □

Proof of sufficiency part. Suppose $0 \leq \epsilon \leq \vec{d}(S_k)$. We prove the k-convexity conditions (10.4.3)–(10.4.5). Since $1 \leq k \leq n-2$, it holds that $\epsilon \leq \vec{d}(S_k) \leq \vec{d}(N\backslash\{n\})$ and so, the first line of Table 10.1 applies stating $g^v(N) = \epsilon$, and for any non-empty coalition S, its gap $g^v(S)$ is either ϵ or $\vec{d}(S)$, whichever is less.

Obviously, for all multi-person coalitions $S \subseteq N$ with $|S| \geq k$, it holds $\epsilon \leq \vec{d}(S_k) \leq \vec{d}(S)$ and so, $v(S) = \vec{d}(S) - \epsilon \geq 0$, and consequently, $g^v(S) = \min[\vec{d}(S), \epsilon] = \epsilon = g^v(N)$. Since the gap $g^v(N)$ is always at the current top level ϵ, it holds $g^v(N) \geq g^v(T)$ for all $T \subseteq N$ with $|T| = k-1$. Therefore, (10.4.3) holds.

Next (10.4.4) holds by the convexity of bankruptcy games.

Finally, by (10.4.4), it holds $g^v(T\cup\{i\}) - g^v(T) \leq g^v(S\cup\{i\}) - g^v(S)$ for all $i \in N, S \subseteq T \subseteq N\backslash\{i\}$, $|T| \leq k-2$, thus, $g^v(S\cup\{i\}) - g^v(S) + g^v(T) \geq g^v(T\cup\{i\})$. By (10.4.3), $g^v(T\cup\{i\}) \geq g^v(N)$ for $|T\cup i| = k$ yielding $g^v(S\cup\{i\}) - g^v(S) + g^v(T) \geq g^v(N)$. This completes the proof of (10.4.5). □

10.7 Concluding Remarks

According to the main Theorem 10.1, the Airport Profit Game is shown to be convex. Consequently, the $1-$convexity property for the Airport Profit Game arises only if the corresponding gap function is fully or partially constant at the level of the cost c_1 of the smallest aircraft. According to the second main Theorem 10.3, the bankruptcy game is k-convex of any type whenever the surplus ϵ is sufficiently small in that the surplus is at most the smallest debt. In case the surplus is strictly more than the smallest debt, then the bankruptcy game is not anymore 1-convex, while the other k-convex notions, $2 \leq k \leq n-2$, remain to be compatible, up to the critical number amounting the sum of the two smallest debts. The 2-convexity property for the bankruptcy game gets lost whenever the surplus strictly exceeds the sum of the two smallest debts, and so on.

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Dongshuang Hou was born on November 10, 1983, in Nanyang city of Henan Province, P.R. China. From 1990 until 2002 he attended primary and secondary school in his hometown. In September 2002, he started to study at Northwestern Polytechnical University in Xi'an. After receiving his Bachelor degree, he went on studying at the same university. He specialized in operations research, in particular game theory. He graduated and received his Master degree after completing a Master's thesis, entitled The solution and properties of Cooperative Games under Matroid, under the supervision of Professor Hao Sun.

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